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A

TREATISE

OF

FLUXIONS.

By ISRAEL LYONS, Junior.

LONDON,
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BRITISH [illegible]

TO

THE REVEREND

ROBERT SMITH, D.D. F.R.S.

and Maſter of TRINITY College,

In the Univerſity of CAMBRIDGE.

REV. SIR,

THE Public, I am perſuaded, will not be ſurpriſed to ſee me preſume to prefix your Name to a Work, which being the firſt Eſſay of a young and unpractiſed Writer, demands a Protection of no common

 Kind

Kind; not that I would be thought to have no better Motive in this Addreſs than a Security to myſelf: The grateful Senſe of the repeated Favours, which I have long experienced from your Humanity, firſt ſuggeſted this Expedient as the only Means of Acknowledgment in my Power; for I know not how to reſiſt the Temptation of telling the Reader, that this Performance owed its firſt rude Beginning to the early Encouragement you were pleaſed to give its Author: Nor ſhould this appear ſtrange, ſince extreme Candour has been ever found the diſtinguiſhing Mark of ſupe-

rior

rior Genius, but it would ill become me to enlarge on a Topic, which however ſlightly touched on, would do Violence to your Delicacy, and, at the ſame time, betray my own officious Vanity. I have the Honour to be

REV. SIR,

Your moſt obliged

And moſt Obedient Servant,

Cambridge,
Nov. 24, 1758.

ISRAEL LYONS, Jun.

PREFACE.

FLUXIONS being found very commodious for the Discovery of new Theorems in Mathematics, I have endeavoured in the following Treatise to facilitate the Knowlege of this Method, by proving it in an easy and concise Manner, and applying it to the different Problems concerning Curve Lines. In which I reject no Quantities as infinitely smaller than the rest, nor suppose different Orders of Infinitesimals and infinitely great Quantities. But consider the Ratio of the Fluxions as the same as that of the contemporaneous Increments, and take Part of the Increment before and Part after the Fluent is arrived at the Term, where we want the Fluxion, since it is not the Increment after, or the Increment before that we want, but at the very instant, which can no otherwise be found but by considering Part of the Increment before and Part after.

In

In finding of Fluents, I have generally followed Mr. COTES's Method by the Meaſures of Ratios, that being the moſt expeditious, and moſt elegant of any.

The common Form for the Variation of Curvature being of little Uſe, becauſe of the Intricacy of the Computations, I have laid down another Form by which it may be found with much more Eaſe.

I have alſo ſolved all Mr. COTES's Problems relating to Curve Lines, and proved thoſe Theorems which he has not demonſtrated.

A

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Of

A TREA-

A TREATISE OF FLUXIONS.

SECTION I.

The direct Method of Fluxions.

Definitions.

1. QUANTITIES are either variable or invariable; the variable are such as are either continually increasing or decreasing by the motion of a point, line, surface, *&c.* the invariable are such as always continue the same. Thus suppose the extremity O, (*fig.* 1.) of the right line O P, to move along the right line A B, so that its other extremity P may trace out the semicircle A P B, then the Lines A O, O P, and the curve A P may be considered as variable quantities, whilst the radius A C is invariable.

Quantities increasing or diminishing in this manner are said to flow, and are called *fluents*.

2. The *fluxion* of a quantity is the velocity or celerity with which that quantity increases or decreases.

3. The invariable quantities are usually represented by the first letters of the alphabet, a, b, c; the variable or fluents by the last, x, y, z; their fluxions by the same letters with a point over them, thus $\dot{x}$ represents the fluxion of the flowing quantity x; $\dot{y}$ that of y. The fluxion of a compound or surd quantity, $\overline{x+y}$ or $\sqrt{a^2+x^2}$ is represented thus, $\dot{\overline{x+y}}$, $\dot{\overline{\sqrt{a^2+x^2}}}$; but the fluxion of a fraction as $\frac{ax}{a+y}$ is thus expressed $\frac{ax}{a+y}\dot{}$.

4. The moments of quantities are the indefinitely small parts, by the addition or substraction of which, in equal particles of time, they are continually increased or diminished.

Proposition I.

The indefinitely small spaces described in equal indefinitely small times are as the velocities.

5. For in uniform motions the spaces are as the velocities, when the times are given, and in accelerated or retarded motions, the increment or decrement of the velocity will decrease as the time decreases, therefore when the time is diminished *ad infinitum*, the difference of the velocities at the beginning and ending of that time will vanish. And the motion may be considered as uniform, and the spaces described in the same particles of time will be as the velocities. *Q. E. D.*

6. *Corollary.* Hence the moments of any flowing quantities are as the Fluxions.

Prop.

PROP. II.

To find the fluxion of the sum or difference of any number of flowing quantities.

7. Thus let it be required to find the Fluxion of $x+y-z+v$. Let the respective moments of x, y, z, v, be X, Y, Z, V, then the next values of x, y, z, v will be $x+X$, $y+Y$, $z+Z$, $v+V$, and substituting these values for x, y, z, v, the next value of $x+y-z+v$ will be $x+X+y+Y-z-Z+v+V$, and the moment will be the difference of these values $X+Y-Z+V$, and since (art. 6.) the moments are as the Fluxions, the Fluxion of $x+y-z+v$ will be $\dot{x}+\dot{y}-\dot{z}+\dot{v}$. Whence to find the Fluxions of quantities connected with the signs $+$ and $-$, we have this rule.

Take the fluxions of every one of the quantities, and connect them together, with the signs of their respective fluents, and the sum will be the fluxion required.

8. *Corol:* Suppose the quantities x, y, z, v, &c. to be equal to each other, then the fluxion of $2x$ will be $\dot{x}+\dot{x}=2\dot{x}$, and the fluxion of $3x=3\dot{x}$; and in general the fluxion of $ax=a\dot{x}$.

PROP. III.

If two flowing quantities x, *and* y, *are always in the same ratio, as* m, *to* n, *to each other, I say their fluxions are to each other in the same ratio.*

9. For since $x:y::m:n$; $x=\frac{m}{n}\times y$ and substituting $x+X$ for x and $y+Y$ for y, as in Art. 7:

we have $x+X=\frac{m}{n}\times y+\frac{m}{n}\times Y$. Whence $X=\frac{m}{n}\times Y$, and $\frac{m}{n}=\frac{X}{Y}=$ (by art. 6.) $\frac{\dot{x}}{\dot{y}}$, therefore $\dot{x}:\dot{y}::m:n$. Q. E. D.

10. *Corol.* 1. Since $\frac{\dot{x}}{\dot{y}}=\frac{m}{n}=\frac{x}{y}$ we have $\frac{\dot{x}}{x}=\frac{\dot{y}}{y}$ and $\dot{y}=\frac{y\dot{x}}{x}$.

Corol. 2. If the fluxions are in a given ratio to each other, the fluents generated in the ſame time, will be to each other in the ſame ratio.

Prop. IV.

If two flowing quantities x *and* y *are to each other in a given ratio, I ſay the fluxion of the rectangle* xy *is equal to* $2y\dot{x}$.

11. Suppoſe $xy=z$, and let its increment be Z, and let x be increaſed by $\frac{1}{2}X$, and y by $\frac{1}{2}Y$ or $\frac{yX}{2x}$, then the area z will become $\overline{x+\frac{1}{2}X}\times\overline{y+\frac{yX}{2x}}=xy+yX+\frac{yX^2}{4x}$. And if x is diminiſhed by $\frac{1}{2}X$, and y by $\frac{1}{2}Y$, the area z will become $\overline{x-\frac{1}{2}X}\times\overline{y-\frac{yX}{2x}}=xy-yX+\frac{yX^2}{4x}$. Therefore whilſt x flows from $x-\frac{1}{2}X$ to $x+\frac{1}{2}X$, the area z flows from $xy-yX+\frac{yX^2}{4x}$ to $xy+yX+\frac{yX^2}{4x}$, and its increment $Z=2yX$. Therefore $2y=\frac{Z}{X}=\frac{\dot{z}}{\dot{x}}$, and $\dot{z}=2y\dot{x}$. Q. E. D.

12. *Corol.*

12. *Corol.* If $x = y$, the fluxion of x^2 is $2x\dot{x}$. In the same manner the fluxion of $\overline{x+y}^2$ is $2 \times \overline{x+y} \times \overline{\dot{x}+\dot{y}} = 2 \times \overline{x\dot{x}+y\dot{x}+x\dot{y}+y\dot{y}}$.

PROP. V.

The fluxion of any rectangle xy is equal $\dot{x}y + x\dot{y}$.

13. The fluxion of $\overline{x+y}^2$, or $x^2+2xy+y^2$ is (by art. 12.) equal to $2 \times \overline{x\dot{x}+\dot{x}y+x\dot{y}+y\dot{y}}$; it is also equal (by art. 7 and 12.) to $2x\dot{x}+2y\dot{y}+\overline{2xy}$, therefore $2x\dot{x}+2\dot{x}y+2x\dot{y}+2y\dot{y} = 2x\dot{x}+2y\dot{y}+\overline{2xy}$, and $\overline{2xy} = 2\dot{x}y+2x\dot{y}$; therefore the fluxion of xy is $\dot{x}y+x\dot{y}$. Q. E. D.

14. *Corol.* 1. If $x = \frac{z}{y}$, then $xy = z$, and by this proposition $\dot{z} = \dot{x}y + x\dot{y}$. Therefore $\dot{x}y = \dot{z} - x\dot{y} = \dot{z} - \frac{z\dot{y}}{y}$, and $\dot{x} = \frac{\dot{z}y - z\dot{y}}{y^2}$ equal to the fluxion of the fraction $\frac{z}{y}$.

15. *Corol.* 2. Suppose $t = yz$, then the fluxion of xyz or xt is $\dot{x}t+x\dot{t} = \dot{x}yz+x\dot{t}$, but $\dot{t} = \dot{y}z+y\dot{z}$, therefore the fluxion of xyz is $\dot{x}yz+x\dot{y}z+xy\dot{z}$. In the same manner the fluxion of $vxyz$ or vxt is $\dot{v}xt+v\dot{x}t+vx\dot{t} = \dot{v}xyz+v\dot{x}yz+vx\dot{y}z+vxy\dot{z}$.

16. *Corol.* 3. If in the last corollary we suppose $x = y = z = v$, then the fluxion of x^3 is $3x^2\dot{x}$, and the fluxion of x^4 is $4x^3\dot{x} = 4x^{4-1}\dot{x}$, and if n be any affirmative integral number, the fluxion of x^n is $nx^{n-1}\dot{x}$.

17. *Corol.* 4. Suppose $x^{-n} = \frac{1}{x^n} = z$, then $1 = zx^n$ and (by art. 16 and 13.) $\dot{z}x^n + nzx^{n-1}\dot{x} = 0$. Therefore $x^n\dot{z} = -nzx^{n-1}\dot{x} = -nx^{-1}\dot{x}$, and $\dot{z} =$

$\dot{z} = -nx^{-n-1}\dot{x} =$ the fluxion of x^{-n} or $\frac{1}{x^n}$.

18. *Corol.* 5. Suppose $x^{\frac{1}{n}} = z$, then $x = z^n$, and $\dot{x} = nz^{n-1}\dot{z} = n\dot{z}x^{1-\frac{1}{n}}$ and $\dot{z} = \frac{1}{n}x^{\frac{1}{n}-1}\dot{x}$, the fluxion of $x^{\frac{1}{n}}$.

19. *Corol.* 6. Suppose $x^{\frac{m}{n}} = z$, then $z^n = x^m$, and $mx^{m-1}\dot{x} = nz^{n-1}\dot{z} = n\dot{z}x^{m-\frac{m}{n}}$, therefore $\dot{z} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$, thefluxion of $x^{\frac{m}{n}}$.

20. From the four last articles it appears, that if n be any number, integral or fractional, affirmative or negative, the fluxion of x^n will be $nx^{n-1}\dot{x}$. Thus the fluxion of $x^{\frac{1}{3}}$ or $\sqrt[3]{x}$ is $\frac{1}{3}x^{-\frac{2}{3}}\dot{x}$, and the fluxion of $\overline{x^2+y^2}|^{\frac{1}{2}}$ is $\frac{1}{2}\times\overline{x^2+y^2}|^{-\frac{1}{2}} \times \overline{2x\dot{x}+2y\dot{y}} = \frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}$ and the flux. of $\overline{e+fx^n}|^{\lambda}$ is $\lambda\times\overline{e+fx^n}|^{\lambda-1}\times nfx^{n-1}\dot{x} = \lambda nf\dot{x}\times x^{n-1}\times\overline{e+fx^n}|^{\lambda-1}$.

21. *Corol.* 7. If the product $vxyz$ is to the product $VXYZ$, as m to n; then $nvxyz = mVXYZ$, and by *Corol.* 2. $n\dot{v}xyz + nv\dot{x}yz + nvx\dot{y}z + nvxy\dot{z} = m\dot{V}XYZ + mV\dot{X}YZ + mVX\dot{Y}Z + mVXY\dot{Z}$, and dividing one side of the equation by $nvxyz$, and the other side by $mVXYZ$, which is equal to it, we shall have $\frac{\dot{v}}{v}+\frac{\dot{x}}{x}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z} = \frac{\dot{V}}{V}+\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}$.

PROP.

Prop. VI.

Having the relation of the fluents, to find the relation of the Fluxions.

22. Let all the terms of the equation expressing the relation of the fluents, be brought to one Side, and made equal to nothing, then find the fluxion of every term, by the preceeding articles, and make the sum of all the fluxions equal 0, which equation will express the relation of the fluxions.

Example I.

23. Let the relation be $x^3 - ax^2 = y^3 - axy$, then bringing all the terms to one side, $x^3 - ax^2 + axy - y^3 = 0$. Now $\dot{\overline{x^3}} = 3x^2\dot{x}$ (art 20.) $\dot{\overline{-ax^2}} = -2ax\dot{x}$ (art. 12.) $\dot{\overline{axy}} = a\dot{x}y + ax\dot{y}$ (art. 13.) and $\dot{\overline{-y^3}} = -3y^2\dot{y}$ (art. 20.) Therefore making the sum of all these fluxions $= 0$, the relation of the fluxions is $3x^2\dot{x} - 2ax\dot{x} + a\dot{x}y + ax\dot{y} - 3y^2\dot{y} = 0$.

24. To find a more general expression for the relation of the fluxions, multiply the equation by $x^m y^n$ (m and n being any two numbers) then $x^{m+3} y^n - a x^{m+2} y^n + a x^{m+1} y^{n+1} - x^m y^{n+3} = 0$, and the relation of the fluxions is $\overline{m+3} \times x^{m+2} y^n \dot{x} + n x^{m+3} y^{n-1} \dot{y} - \overline{m+2} \times a x^{m+1} y^n \dot{x} - n a x^{m+2} y^{n-1} \dot{y} + \overline{m+1} \times a x^m y^{n+1} \dot{x} + \overline{n+1} \times a x^{m+1} y^n \dot{y} - m x^{m-1} y^{n+3} \dot{x} - \overline{n+3} \times x^m y^{n+2} = 0$, divide by $x^m y^n$, and the equation will be $\overline{m+3} \times x^2 \dot{x} + n x^3 y^{-1} \dot{y} - \overline{m+2} \times a x \dot{x} - n a x^2 y^{-1} \dot{y} + \overline{m+1} \times a y \dot{x} + \overline{n+1} \times a x \dot{y} - m x^{-1} y^3 \dot{x} - \overline{n+3} \times y^2 \dot{y} = 0$, which expresses the relation between $\dot{x}$ and $\dot{y}$, let m and n be what they will; as suppose

 $m = 0$,

$m=0$, $n=0$, then $3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0$, the ſame equation as was found before. If $m=-1$ and $n=0$, then $2x^2\dot{x} - ax\dot{x} + ax\dot{y} + x^{-1}y^3\dot{x} - 3y^2\dot{y} = 0$. And if $m=1$, $n=-1$, the relation will be $4x^2\dot{x} - x^3y^{-1}\dot{y} - 3ax\dot{x} + ax^2y^{-1}\dot{y} + 2ay\dot{x} - y^3x^{-1}\dot{x} - 2y^2\dot{y} = 0$.

Theſe different expreſſions are no others than ariſe from the different forms of the equation for in the general expreſſion $\frac{\dot{x}}{\dot{y}} =$

$$\frac{\overline{n+3}\times y^2 - \overline{n+1}\times ax + nax^2y^{-1} - nx^3y^{-1}}{\overline{m+3}\times x^2 - \overline{m+2}\times ax + \overline{m+1}\times ay - mx^{-1}y^3} = \frac{ny^2 + 3y^2 - nax - ax + nax^2y^{-1} - nx^3y^{-1}}{mx^2 + 3x^2 - max - 2ax + may + ay - mx^{-1}y^3} = \frac{3y^2 - ax + ny^{-1}\times\overline{y^3 - axy + ax^2 - x^3}}{3x^2 - 2ax + ay + mx^{-1}\times\overline{x^3 - ax^2 + axy - y^3}} =$$

$\frac{3y^2 - ax}{3x^2 - 2ax + ay}$ becauſe $x^3 - ax^2 + axy - y^3 = 0$; whence it appears, that all theſe different equations expreſſing the relation of the fluxions are the ſame in reality, their difference ariſing only from the ſeveral forms the equation may put on.

25. By this general expreſſion we may always find the moſt ſimple equation ſhewing the relation between the fluxions $\dot{x}$ and $\dot{y}$, by ſubſtituting for m and n, ſuch numbers as will make the greateſt deſtruction of terms, which numbers may be found by conſidering which of the terms m, $m+1$, $m+2$, occurs ofteneſt, and making that term $=0$, you may determine m, and in the ſame manner you may find n. Thus in the laſt example m, $m+1$, $m+2$, $m+3$, occur each of them but once, therefore

fore make $m=0$, -1, -2, or -3. But of n, $n+1$, it appears that n occurs ofteneſt, therefore make $n=0$, and the ſimpleſt equations of the fluxions are,

Firſt, $m=0$, $n=0$. . . . $3x^2-2ax\dot{x}+ay\dot{x}+ax\dot{y}-3y^2\dot{y}=0$.

Second, $m=-1$, $n=0$, $2x^2\dot{x}-ax\dot{x}+ax\dot{y}+y^3x^{-1}\dot{x}-3y^2\dot{y}=0$.

Third, $m=-2$, $n=0$. $x^2\dot{x}-ay\dot{x}+ax\dot{y}+2y^3x^{-1}\dot{x}-3y^2\dot{y}=0$.

Fourth, $m=-3$, $n=0$. $ax\dot{x}-2ay\dot{x}+ax\dot{y}+3y^3x^{-1}\dot{x}-3y^2\dot{y}=0$.

Example II.

26. Let the relation be $x^3-a^2x+xy^2-a^2y=0$, multiply by x^my^n, then $x^{m+3}y^n-a^2x^{m+1}y^n+x^{m+1}y^{n+2}-a^2x^my^{n+1}=0$, and taking the fluxions $\overline{m+3}\times x^{m+2}y^n\dot{x}+nx^{m+3}y^{n-1}\dot{y}-\overline{m+1}\times a^2x^my^n\dot{x}-na^2x^{m+1}y^{n-1}\dot{y}+\overline{m+1}\times x^my^{n+2}\dot{x}+\overline{n+2}\times x^{m+1}y^{n+1}\dot{y}-ma^2x^{m-1}y^{n+1}\dot{x}-\overline{n+1}\times a^2x^my^n\dot{y}=0$, divide by x^my^n, then $\overline{m+3}\times x^2\dot{x}+nx^3y^{-1}\dot{y}-\overline{m+1}\times a^2\dot{x}-na^2xy^{-1}\dot{y}+\overline{m+1}\times y^2\dot{x}+\overline{n+2}\times xy\dot{y}-ma^2x^{-1}y\dot{x}-\overline{n+1}\times a^2\dot{y}=0$, the general equation expreſſing the relation of $\dot{x}$ and $\dot{y}$. And to find the moſt ſimple equation, we muſt make $m+1=0$, or $m=-1$, and $n=0$; in which caſe, the relation will be $2x^2\dot{x}+2xy\dot{y}+a^2x^{-1}y\dot{x}-a^2\dot{y}=0$.

27. To find the ſimpleſt equation of the fluxions it is not neceſſary to find the general fluxional equation, it being ſufficient to multiply the equation by $x^my^nz^p$, &c. and to obſerve which of the indexes m, $m+1$, $m+2$; n, $n+1$; p, $p+1$, *&c.* occurs

occurs ofteneſt, and by putting thoſe indexes $=0$, you may determine m, n, p, &c. and then ſubſtituting their values, or which is the ſame thing, divide the equation by thoſe powers of x, y, z, &c. which occur ofteneſt, then find the equation of the fluxions, which will be the moſt ſimple one.

Example III.

28. Let the equation be $xy - xz + yz = 0$, multiply by $x^m y^n z^p$, then $x^{m+1} y^{n+1} z^p - x^{m+1} y^n z^{p+1} + x^m y^{n+1} z^{p+1} = 0$, therefore make $m+1=0$, $n+1=0$, $p+1=0$; or $m=-1$, $n=-1$, $p=-1$, and ſubſtituting theſe values, $z^{-1} - y^{-1} + x^{-1} = 0$, and taking the fluxions $\frac{-\dot{z}}{zz} + \frac{\dot{y}}{yy} - \frac{\dot{x}}{xx} = 0$, which is the ſimpleſt equation ſhewing the relation between $\dot{x}$, $\dot{y}$ and $\dot{z}$.

Example IV.

29. Let the equation be $xx + x\sqrt{aa - yy} - \frac{a^3}{a+y} = 0$. To take away the ſurd quantity $\sqrt{aa-yy}$, and the compound fraction $\frac{a^3}{a+y}$, ſuppoſe $aa - yy = zz$, and $\frac{a^3}{a+y} = v$, then the equation will be $xx + xz - v = 0$, and taking the fluxions $2x\dot{x} + \dot{x}z + x\dot{z} - \dot{v} = 0$. But by the equations $aa - yy = zz$ and $v = \frac{a^3}{a+y}$ or $av + vy = a^3$, we have $z\dot{z} = -y\dot{y}$, and $\dot{z} = \frac{-y\dot{y}}{\sqrt{aa-yy}}$, and $a\dot{v} + \dot{v}y + v\dot{y} = 0$, or $\dot{v} = \frac{-a^3\dot{y}}{\overline{a+y}|^2}$, and ſubſtituting theſe

values

values of $\dot{z}$ and $\dot{v}$, we have $2x\dot{x} + \dot{x}\sqrt{aa-yy}$ $\dfrac{-xy\dot{y}}{\sqrt{aa-yy}} + \dfrac{a^3\dot{y}}{\overline{a+y}^2} = 0$, which shews the relation of the fluxions $\dot{x}$ and $\dot{y}$.

Of Second, Third and Higher FLUXIONS.

30. IN the same manner as the quantities $x, y, z,$ are conceived to flow, and to have their fluxions, so may the quantities $\dot{x}, \dot{y}, \dot{z},$ be supposed to be variable, and therefore have their fluxions, which are thus represented $\ddot{x}, \ddot{y}, \ddot{z},$ and are called the second fluxions of x, y, z: these quantities also have their fluxions $\dddot{x}, \dddot{y}, \dddot{z},$ which are the third fluxions of x, y, z. These also have their fluxions $\ddddot{x}, \ddddot{y}, \ddddot{z},$ which are the fourth fluxions of x, y, z; and so on.

31. The relation of these higher fluxions are derived from the equation expressing the relation of the preceeding orders of fluxions, in the same manner as we derived the relation of the first fluxions from the relation of the fluents.

Example I.

32. To find the relation of the third fluxions $\dddot{x}$ and $\dddot{y}$, when the relation of the fluents is $xx - ay = 0$. The relation of the first fluxions is $2x\dot{x} - a\dot{y} = 0$. Now in this equation, as we have three flowing quantities x, $\dot{x}$ and $\dot{y}$, the equation of the second fluxions will be $2\dot{x}\dot{x} + 2x\ddot{x} - a\ddot{y} = 0$. Here we have four flowing quantities x, $\dot{x}$, $\ddot{x}$, and $\ddot{y}$, therefore the equation of the third fluxions will

will be $4\dot{x}\ddot{x}+2\dot{x}\ddot{x}+2x\dddot{x}-a\dddot{y}=6\dot{x}\ddot{x}+2x\dddot{x}-a\dddot{y}=0$.

33. Since it is only the relation of theſe fluxions that we are ſeeking, we may ſuppoſe ſome quantity, as x to flow uniformly, and then its fluxion will be an invariable quantity, and the quantity x will have no ſecond fluxion, and conſequently no third or higher fluxion, which will make the fluxional equation much more ſimple; thus in the laſt example, the ſecond fluxional equation was $2\dot{x}\dot{x}+2x\ddot{x}-a\ddot{y}=0$, put $\ddot{x}=0$, and the equation will be $2\dot{x}\dot{x}-a\ddot{y}=0$, again the third fluxional equation was $6\dot{x}\ddot{x}+2x\dddot{x}-a\dddot{y}=0$; which by putting $\ddot{x}=0$, and $\dddot{x}=0$, will become $-a\dddot{y}=0$, or $\dddot{y}=0$, and therefore the quantity y will have no third fluxion.

Example II.

34. Let the relation of the fluents be $x^m-y=0$, then the firſt fluxional equation is $mx^{m-1}\dot{x}-\dot{y}=0$. Here we have two variable quantities x and $\dot{y}$, ſuppoſing $\dot{x}$ invariable (art. 33.) Therefore the ſecond fluxional equation is $m\times\overline{m-1}\times x^{m-2}\dot{x}^2-\ddot{y}=0$. In like manner, the third fluxional equation is $m\times\overline{m-1}\times\overline{m-2}\times x^{m-3}\dot{x}^3-\dddot{y}=0$, and the fourth $m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times x^{m-4}\dot{x}^4-\ddddot{y}=0$, and ſo on. If the index m is an affirmative integral number, this ſeries of fluxional equations will at laſt break off, but if m is a fractional or negative number, it will run on *ad infinitum*, and will never terminate.

Example

Example III.

35. Let the relation be $xy-z=0$, then the first fluxional equations is $\dot{x}y+x\dot{y}-\dot{z}=0$, and the second supposing $\dot{x}$ invariable is $\dot{x}\dot{y}+\dot{x}\dot{y}+x\ddot{y}-\ddot{z}=2\dot{x}\dot{y}+x\ddot{y}-\ddot{z}=0$, and third is $3\dot{x}\ddot{y}+x\dddot{y}-\dddot{z}=0$, in the same manner the fourth is $4\dot{x}\dddot{y}+x\ddddot{y}-\ddddot{z}=0$.

36. In the same manner, we may find the fluxion of a quantity composed of fluxions. As for example, to find the fluxion of $\frac{\dot{y}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}}$ supposing $\dot{x}$ invariable, let $\frac{\dot{y}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}}=z$, then then $\dot{y}^2\dot{x}^2+\dot{y}^4=\dot{x}^2z^2$, and taking the fluxions $\dot{x}^2\dot{y}\ddot{y}+2\dot{y}^3\ddot{y}=\dot{x}^2z\dot{z}$, and $\dot{z}=\frac{\dot{x}^2\dot{y}\ddot{y}+2\dot{y}^3\ddot{y}}{\dot{x}^2z}=$ $\frac{\dot{x}^2\dot{y}\ddot{y}+2\dot{y}^3\ddot{y}}{\dot{x}\dot{y}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}=\frac{\dot{x}^2\ddot{y}+2\dot{y}^2\ddot{y}}{\dot{x}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$.

SECT. II.

The inverse Method of Fluxions.

THE fluent of any quantity as $x^m\dot{x}$ is represented thus $|\overline{x^m\dot{x}}|$, in the same manner $|\underline{\overline{\dot{x}y}}|$ represents the fluent of $\dot{x}y$.

PROP. VII.

To find the fluent of the fluxion $ax^m\dot{x}$.

37. By art 20. it appears that the fluxion of the fluent x^{m+1} is $\overline{m+1}\times x^m\dot{x}$, and this last fluxion

fluxion is to the proposed fluxion as $m+1$ to a, therefore by *Cor.* 2 *Prop.* 3. the fluents are in the same ratio, that is $m+1 : a :: x^{m+1} : \overline{|a x^m \dot{x}|}$ and

$\overline{|ax^m\dot{x}|} = \frac{a}{m+1} \times x^{m+1}$. Q. E. I.

Examples.

1. To find the fluent of $x\dot{x}$. Here $a=1$, $m=1$, $m+1=2$, and the fluent is $\frac{a}{m+1} \times x^{m+1} = \frac{1}{2}x^2$.

2. To find the fluent of $2x^{\frac{3}{2}}$, $a=2$, $m=\frac{3}{2}$, $m+1=\frac{5}{2}$ and the fluent is $\frac{4}{5}x^{\frac{5}{2}}$.

3. To find the fluent of $\frac{\dot{x}}{x^2}$. $a=1$, $m=-2$, $m+1=-1$, and the fluent is $-\frac{1}{x}$.

4. To find the fluent of $\frac{2\dot{x}}{\sqrt{x}}$ or $2\dot{x}x^{-\frac{1}{2}}$ here $a=2$, $m=-\frac{1}{2}$, $m+1=\frac{1}{2}$, and the fluent is $4x^{\frac{1}{2}}=4\sqrt{x}$.

38. If the quantity C be invariable, then the fluxion of $x+C$ is $\dot{x}$, since the constant quantity C has no fluxion, therefore $\dot{x}$ will have the fluent $x+C$, as well as x. This constant quantity C is known from the conditions of the fluent, by finding where the fluent vanishes, and by substituting the value of x in that case, and making the result $=0$, you may determine C, as for example suppose the fluent $\frac{1}{2}x^2$ (in Exam. 1. art. 37.) was to vanish, when $x=b$, then $\frac{1}{2}b^2+C=0$, and $C=-\frac{1}{2}b^2$, therefore the correct fluent is $\frac{1}{2}x^2+C$ is $\frac{1}{2}x^2-\frac{1}{2}b^2$.

PROP.

PROP. VIII.

To find the fluent of any number of simple quantities each of which involve but one flowing quantity.

39. It appears by Prop. II. that the fluent of $\dot{x}-\dot{y}+\dot{z}$ is $x-y+z$. Therefore find the fluents of every term by the last proposition, and their sum will be the fluent. *Q. E. I.*

Example I.

To find the fluent of $2x\dot{x}+\frac{3x^2\dot{x}}{a}-\frac{a^3\dot{x}}{x^2}$, which is to vanish, when $x=a$, The fluent of $2x\dot{x}$ is xx, $\left|\overline{\frac{3x^2\dot{x}}{a}}\right|=\frac{x^3}{a}$, and $\left|\overline{\frac{a^3\dot{x}}{x^2}}\right|=\frac{-a^3}{x}$, and the sum of these fluents is the fluent required, $x^2+\frac{x^3}{a}+\frac{a^3}{x}+C$, put $x=a$, then $3a^2+C=0$, and $C=-3a^2$, therefore the corrected fluent is $x^2+\frac{x^3}{a}+\frac{a^3}{x}-3a^2$.

Example II.

To find the fluent of $\frac{x^2\dot{x}}{a}+\frac{a^4\dot{x}}{x^2}-\frac{7x^6\dot{x}}{a^4}$. Now $\left|\overline{\frac{x^2\dot{x}}{a}}\right|=\frac{x^3}{3a}$, $\left|\overline{\frac{a^4\dot{x}}{x^2}}\right|=-\frac{a^4}{x}$, $\left|\overline{\frac{7x^6\dot{x}}{a^4}}\right|=\frac{x^7}{a^4}$, and the fluent is $\frac{x^3}{3a}-\frac{a^4}{x}-\frac{x^7}{a^4}+C$. Suppose it to

vanish

vanish, when $x=a$, then $\frac{a^2}{3}-2a^2+C=0$, and $C=\frac{5}{3}a^2$, therefore the correct fluent is $\frac{x^3}{3a}-\frac{a^4}{x}-\frac{x^7}{a^2}+\frac{5a^2}{3}$.

Example III.

To find the fluent of $x^{\frac{1}{2}}\dot{x}+\frac{a\dot{x}}{x^{\frac{1}{2}}}$. Here $\left|\overline{x^{\frac{1}{2}}\dot{x}}\right|=\frac{2}{3}x^{\frac{3}{2}}$, and $\left|\overline{\frac{a\dot{x}}{x^{\frac{1}{2}}}}\right|=2ax^{\frac{1}{2}}$, therefore the fluent is $\frac{2}{3}x^{\frac{3}{2}}+2ax^{\frac{1}{2}}+C$, suppose it to vanish when $x=\frac{1}{4}a$, then $x^{\frac{1}{2}}=\frac{1}{2}a^{\frac{1}{2}}$ and $\frac{1}{12}a^{\frac{3}{2}}+a^{\frac{3}{2}}-C=0$ and $C=-\frac{13}{12}a^{\frac{3}{2}}$ therefore the correct fluent is $\frac{2}{3}x^{\frac{3}{2}}+2ax^{\frac{1}{2}}-\frac{13}{12}a^{\frac{3}{2}}$.

Example IV.

To find the fluent of $a\dot{x}+3x\dot{x}$, which shall vanish, when $x=0$. Now $\left|\overline{a\dot{x}}\right|=ax$, $\left|\overline{3x\dot{x}}\right|=\frac{3}{2}xx$ and the fluent is $ax+\frac{3}{2}x^2+C$, make $x=0$, then $C=0$, and the correct fluent is $ax+\frac{3}{2}x^2$.

Lemma I.

40. *Let* x *be any variable quantity, susceptible of any value, and let* $A+Bx+Cx^2+Dx^3+$ &c. $=0$, *I say* $A=0$, $B=0$, $C=0$, $D=0$.

For put $x=0$, then $A=0$, and $Bx+Cx^2+Dx^3+$ &c. $=0$. therefore $B+Cx+Dx^2+$ &c. $=0$, whence $B=0$ and $C+Dx=0$, therefore $C=0$ and $D=0$. *Q. E. D.*

41. *Corol.* Hence if $A+Bx+Cx^2+Dx^3+$ &c. be always equal to $a+bx+cx^2+dx^3+$ &c. then

$A=a$

$A=a$, $B=b$, $C=c$, $D=d$, for by transposition

$$\left.\begin{matrix} A+B \\ -a-b \end{matrix}\right\} x \quad \left.\begin{matrix} +C \\ -c \end{matrix}\right\} x^2 \quad \left.\begin{matrix} +D \\ -d \end{matrix}\right\} x^3 + \&c. = 0,$$

therefore by this Lemma $A-a=0$, $B-b=0$, $C-c=0$, $D-d=0$, and $A=a$, $B=b$, $C=c$, $D=d$.

Lemma II.

To find the value of $\overline{a+x}|^m$ *in an infinite series.*

42. It appears from the common algebra, that the exponents of the quantity a will continually decrease, and the exponents of x will continually increase by 1. Therefore suppose $\overline{a+x}|^m = A + AB \times \frac{x}{a} + ABC \times \frac{x^2}{a^2} + ABCD \times \frac{x^3}{a^3} + \&c.$ multiply by $a+x$ then $\overline{a+x}|^{m+1} =$

$$\left.\begin{matrix} Aa + AB \\ +A \end{matrix}\right\} x \quad \left.\begin{matrix} +ABC \\ +AB \end{matrix}\right\} \frac{x^2}{a} \quad \left.\begin{matrix} +ABCD \\ +ABC \end{matrix}\right\} \frac{x^3}{a^2} + \&c.$$

and taking the fluxions, and dividing by $\dot{x}$, we have

$$\overline{m+1} \times \overline{a+x}|^m = \left.\begin{matrix} AB + 2ABC \\ +A + 2AB \end{matrix}\right\} \frac{x}{a} \left.\begin{matrix} +3ABCD \\ +3ABC \end{matrix}\right\} \frac{x^2}{a^2} +$$

&c. But $\overline{a+x}|^m = A + AB \times \frac{x}{a} + ABC \times \frac{x^2}{a^2} + \&c.$

therefore by substraction $\left.\begin{matrix} AB + 2ABC \\ + AB \end{matrix}\right\} \frac{x}{a}$

$$\left.\begin{matrix} +3ABCD \\ +2ABC \end{matrix}\right\} \frac{x^2}{a^2} + \&c. = m \times \overline{a+x}|^m = mA + mAB \times$$

$\frac{x^2}{a^2} + \&c.$ Therefore by *Cor. Lem.* I. $AB = mA$,

$2ABC + AB = mAB$, $3ABCD + 2ABC = mABC$, whence $B = m$, $C = \frac{m-1}{2}$, $D = \frac{m-2}{3}$, and when $x = 0$, $A = a^m$, therefore $\overline{a+x}|^m = a^m + ma^{m-1}x + m.\frac{m-1}{2}.\,a^{m-2}x^2 + m.\frac{m-1}{2}.\frac{m-2}{3}.\,a^{m-3}x^3 +$ &c. By this theorem of Sir I. NEWTON's*, we may find at once any power or root of any binomial, trinomial, &c. without computing the intermediate powers.

Examples.

43. *Exam.* I. To find the fourth power of $a+x$, we have $m=4$, $\frac{m-1}{2} = \frac{3}{2}$, $\frac{m-2}{3} = \frac{2}{3}$, $\frac{m-3}{4} = \frac{1}{4}$, $\frac{m-4}{5} = 0$. And $\overline{a+x}|^4 = a^4 + 4a^3x + 4.\frac{3}{2}.\,a^2x^2 + 4.\frac{3}{2}.\frac{2}{3}.\,ax^3 + 4.\frac{3}{2}.\frac{2}{3}.\frac{1}{4}\,x^4, = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$. Here the series breaks off, and becomes finite, which will always happen when m is an integral affirmative number. In other cases we shall have an infinite series, which will converge the faster, the less x is in respect of a.

Example II.

To divide 1 by $a+x$, or to find the value of $\overline{a+x}|^{-1}$. We have $m = -1$, $\frac{m-1}{2} = -1$, $\frac{m-2}{3} = -1$, and $\frac{1}{a+x} = a^{-1} - a^{-2}x + a^{-3}x^2 - a^{-4}x^3 +$ &c. $= \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} +$ &c.

* Letter to Mr. *Oldenburg*. 13 *Jun*. 1676.

Example

Example III.

To find the value of $\frac{1}{\overline{a+x}\rvert^{n}}$ or $\overline{a+x}\rvert^{-n}$ we have, $m=-n$, $\frac{m-1}{2}=-\frac{n+1}{2}$, $\frac{m-2}{3}=-\frac{n+2}{3}$, and $\overline{a+x}\rvert^{-n}=a^{n}-na^{-n-1}x+n\times\frac{n+1}{2}a^{-n-2}x^{2}$ $-n.\frac{n+1}{2}.\frac{n+2}{3}.a^{-n-3}x^{3}+\&c.=\frac{1}{a^{n}}-\frac{nx}{a^{n+1}}+\frac{n.\,n+1.\,x^{2}}{2a^{n+2}}-\frac{n.\,n+1.\,n+2.\,x^{3}}{2.\,3\,a^{n+3}}+\&c.$

Example IV.

To extract the square root of $a+x$, or to find the value of $\overline{a+x}\rvert^{\frac{1}{2}}$, we have $m=\frac{1}{2}$, $\frac{m-1}{2}=\frac{-1}{4}$, $\frac{m-2}{3}=\frac{-3}{6}=\frac{-1}{2}$, $\frac{m-3}{4}=\frac{-5}{8}$, $\frac{m-4}{5}=\frac{-7}{10}$ and $\sqrt{a+x}=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}x-\frac{1.\,1}{2.\,4}a^{-\frac{3}{2}}x^{2}+\frac{1.\,1.\,1}{2.\,4.\,2}a^{-\frac{5}{2}}x^{3}-\frac{1.\,1}{2.\,4.}\frac{1.\,5}{2.\,8}a^{-\frac{7}{2}}x^{4}+\&c.=a^{\frac{1}{2}}+\frac{x}{2a^{\frac{1}{2}}}-\frac{x^{2}}{8a^{\frac{3}{2}}}+\frac{x^{3}}{16a^{\frac{5}{2}}}-\frac{5x^{4}}{128a^{\frac{7}{2}}}+\&c.$ In the same manner $\sqrt{r^{2}+z^{2}}=r+\frac{z^{2}}{2r}-\frac{z^{4}}{8r^{3}}+\frac{z^{6}}{16r^{5}}-\frac{5z^{8}}{128r^{7}}+\&c.$ and $\sqrt{r^{2}-z^{2}}=r-\frac{z^{2}}{2r}-\frac{z^{4}}{8r^{3}}-\frac{z^{6}}{16r^{5}}-\frac{5z^{8}}{128r^{7}}-\&c.$

Example V.

To find $\overline{a+x}|^{\frac{n}{p}}$ in an infinite series, $m=\frac{n}{p}$, $\frac{m-1}{2}=\frac{n-p}{2}$, $\frac{m-2}{3}=\frac{n-2p}{3p}$, and $\overline{a+x}|^{\frac{n}{p}}=a^{\frac{n}{p}}$ $+\frac{n}{p}.\ a^{\frac{n-p}{p}}x+\ \frac{n}{p}.\ \frac{n-p}{2p}.\ a^{\frac{n-2p}{p}}x^2+\ \frac{n}{p}.\ \frac{n-p}{2p}.$ $\frac{n-2p}{3p}.\ a^{\frac{n-3p}{p}}x^3+$ &c.

Example VI.

To find the square of the trinomial $a+b+c$, we must consider $a+b+c$ as a binomial by making $a=a$, and $x=b+c$, then $m=2$, $\frac{m-1}{2}=\frac{1}{2}$ and $\overline{a+b+c}|^2=a^2+2a\times\overline{b+c}+\overline{b+c}|^2=a^2+2ab+$ $2ac+b^2+2bc+c^2$.

44. *Corol*, Put $\frac{x}{a+x}=y$, then $x=\frac{ay}{1-y}$ and $a+x=\frac{a}{1-y}$ and $\overline{a+x}|^m=a^m\times\overline{1-y}|^{-m}=$ (by *E*. 3.)

$a^m\times 1+my+\frac{m.\ \overline{m+1}}{2}\times y^2+\frac{m.\overline{m+1}.\overline{m+2}}{2.\ 3}\times y^3$ $+$&c. $=a^m+\frac{m\,a^m x}{a+x}+\frac{m.\ \overline{m+1}}{2}\times\frac{a^m\,x^2}{\overline{a+x}|^2}+$ $\frac{m.\ \overline{m+1}.\ \overline{m+2}}{2.\ 3}\times\frac{a^m\,x^3}{\overline{a+x}|^3}+$ &c. Which is Mr. COLSON's Theorem.

45. To find any power m of the multinomial $a+b+c+d+$ &c. make $a=a$, $x=b+c+d+$&c. Then

Then $\overline{a+b+c+d+\&c.}|^m = a^m + m a^{m-1} \times \overline{b+c+d+\&c.} + m \times \frac{m-1}{2} \times a^{m-2} \times \overline{b+c+d}|^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} \times \overline{b+c+d}|^3 + \&c.$ But $\overline{b+c+d}|^2 = b^2 + 2bc + \&c.$ and $\overline{b+c+d}|^3 = b^3 + \&c.$ Therefore $\overline{a+b+c+d+\&c.}|^m =$

$$a^m + m a^{m-1} b + m a^{m-1} c + m a^{m-1} d + \&c.$$

$$+ m \times \frac{m-1}{2} \times a^{m-2} b^2 + m \times \frac{m-1}{1} \times a^{m-2} bc + \&c.$$

$$+ m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \&c.$$

Which is Mr. DE MOIVRE's theorem for raising a multinomial to any power m *.

PROP. IX.

To find the fluent of a compound or surd quantity, which involves but one flowing quantity.

46. Reduce the quantity to an infinite series by *Lemma* II. then find the fluents of every term, by *Prop.* VII. and their sum will be the fluent. Q. E. I.

Examples.

Example I. To find the fluent of $\frac{a\dot{x}}{a+x}$, this quantity is equal to $a\dot{x} \times \overline{a+x}|^{-1} =$ (by Exam. II. art. 43.) $a\dot{x} \times \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \&c. = \dot{x} - \frac{x\dot{x}}{a} + \frac{x^2\dot{x}}{a^2} - \frac{x^3\dot{x}}{a^3} + \&c.$ And the fluent is $x - \frac{x^2}{2a} + \frac{x^3}{3a^2} - \frac{x^4}{4a^3} + \&c.$

* Miscel. Analyt. p. 87.

Example II.

To find the fluent of $\dot{x}\sqrt{rr-xx}=\dot{x}\times\overline{r^2-x^2}|^{\frac{1}{2}}$ $=$ (by Exam. 4. art. 43.) $r\dot{x}-\frac{x^2\dot{x}}{2r}-\frac{x^4\dot{x}}{8r^3}-\frac{x^6\dot{x}}{16r^5}-$ &c. whose fluent is $rx-\frac{x^3}{6r}-\frac{x^5}{40r^3}-\frac{x^7}{112r^5}-$ &c.

Example III.

To find the fluent of $\frac{\dot{x}}{e+fx+gx^2}=\dot{x}\times\overline{e+fx+gx^2}|^{-1}$. In art. 45. make $a=e$, $b=fx$, $c=gx^2$, $d=0$, $m=-1$, then $\dot{x}\times\overline{e+fx+gx^2}|^{-1}$ $=\frac{\dot{x}}{e}-\frac{fx\dot{x}}{e^2}+\frac{f^2-eg}{e^3}\times x^2\dot{x}+\frac{2efg-f^3}{e^4}x^3\dot{x}$ $+$&c. and the fluent is $\frac{x}{e}-\frac{fx^2}{2e^2}+\frac{f^2-eg}{3e^3}\times x^3+\frac{2efg-f^3}{4e^4}\times x^4+$ &c.

47. In general to find the fluent of $\dot{x}x^{\theta-1}\times\overline{e+fx^n}|^{\lambda}$ by Lem. 2, we have $\overline{e+fx^n}|^{\lambda}=e^{\lambda}+\lambda.\, e^{\lambda-1}fx^n+\frac{\lambda.\,\overline{\lambda-1}}{2}.\, e^{\lambda-2}f^2x^{2n}+\frac{\lambda.\,\overline{\lambda-1}.\,\overline{\lambda-2}}{2.\;3}.\, e^{\lambda-3}f^3x^{3n}+$ &c. and $\dot{x}x^{\theta-1}\times\overline{e+fx^n}|^{\lambda}=e^{\lambda}\dot{x}x^{\theta-1}+\lambda.\, e^{\lambda-1}f\dot{x}x^{\theta+n-1}+\frac{\lambda.\overline{\lambda-1}}{2}.\, e^{\lambda-2}f^2\dot{x}x^{\theta+2n-1}$ $+\frac{\lambda.\overline{\lambda-1}.\overline{\lambda-2}}{2\,.\,3}.\, e^{\lambda-3}f^3\dot{x}x^{\theta+3n-1}+$ &c.

whose

whose fluent is $\frac{1}{\theta}e^{\lambda}x^{\theta}+\frac{\lambda}{\theta+\eta}.\ e^{\lambda-1}fx^{\theta+\eta}+$ $\frac{\lambda.\lambda-1}{2.\theta+2\eta}.\ e^{\lambda-2}f^{2}x^{\theta+2\eta}+\frac{\lambda^{?}.\lambda-1.\lambda-2}{2.3.\theta+3\eta}.\ e^{\lambda-3}$ $f^{3}x^{\theta+3\eta}+$ &c. Where if we put A, B, C, &c. for the several terms of the series, A for the first $\frac{1}{\theta}e^{\lambda}x^{\theta}$, B for the second $\frac{\lambda}{\theta+\eta}.\ e^{\lambda-1}fx^{\theta+\eta}$ and so on, the series will become $\frac{1}{\theta}e^{\lambda}x^{\theta}+\frac{\lambda.\theta}{\theta+\eta}\text{A}\frac{fx^{\eta}}{e}$ $+\frac{\lambda-1.\theta+\eta}{2.\theta+2\eta}\text{B}\frac{fx^{\eta}}{e}+\frac{\lambda-2.\theta+2\eta}{3.\theta+3\eta}\text{C}\frac{fx^{\eta}}{e}$, &c. which is a general Theorem for finding the fluent of any quantity of this form $\dot{x}x^{\theta-1}\times\overline{e+fx^{\eta}}|^{\lambda}$.

Example.

To find the fluent of $\dot{x}\sqrt{r^{2}-x^{2}}$, we have $\theta-1=0$ and $\theta=1$, $e=r^{2}$, $f=-1$, $\eta=2$, $\lambda=\frac{1}{2}$, and the fluent is $rx-\frac{\frac{1}{2}}{3}\text{A}\frac{x^{2}}{r^{2}}+\frac{\frac{1}{2}\cdot 3}{2\cdot 5}\text{B}\frac{x^{2}}{r^{2}}+\frac{\frac{3}{2}\cdot 5}{3\cdot 7}$ $\text{C}\frac{x^{2}}{r^{2}}$, &c. $=rx-\frac{1}{2\cdot 3}\text{A}\frac{x^{2}}{r^{2}}+\frac{1\cdot 3}{4\cdot 5}\text{B}\frac{x^{2}}{r^{2}}+\frac{3\cdot 5}{6\cdot 7}$ $\text{C}\frac{x^{2}}{r^{2}}+$ &c. $=rx-\frac{x^{3}}{6r}-\frac{x^{5}}{40r^{3}}-\frac{x^{7}}{112r^{5}}-$ &c. as we found before *Exam.* 2, *Art.* 46.

48. Sometimes the fluent of a surd quantity may be expressed by another surd, thus the fluent of $\dot{x}x\sqrt{r^{2}-x^{2}}$ is $-\frac{1}{3}\times\overline{r^{2}-x^{2}}|^{\frac{3}{2}}$, and the fluent of

$\dot{x}x^{\eta-1}\times\overline{e+fx^{\eta}}|^{\lambda-1}$ is $\frac{1}{\lambda\eta f}\times\overline{e+fx^{\eta}}|^{\lambda}$ (*art.* 20.)

Prop. X.

To find the fluent of $\dot{x}x^{\theta-1}\times\overline{e+fx^{\eta}}|^{\lambda-1}$.

49. Put $\frac{x^{\eta}}{e+fx^{\eta}}=z$, then $x^{\eta}=\frac{ez}{1-fz}$, and $e+fx^{\eta}=\frac{e}{1-fz}$. Also $\frac{1}{z}=\frac{e}{x^{\eta}}+f$, whence taking the Fluxions $\frac{\dot{z}}{zz}=\frac{\eta e\dot{x}}{x^{\eta+1}}$, therefore $\dot{x}x^{-1}$ $=\frac{\dot{z}x^{\eta}}{\eta ez^{2}}=\frac{e\dot{z}}{\eta ez\times\overline{1-fz}}=\frac{\dot{z}z^{-1}}{\eta\times\overline{1-fz}}$, also we have $x^{\theta}=\frac{e^{\frac{\theta}{\eta}}z^{\frac{\theta}{\eta}}}{\overline{1-fz}|^{\frac{\theta}{\eta}}}$, and since $e+fx^{\eta}=\frac{e}{1-fz}$, consequently $\overline{e+fx^{\eta}}|^{\lambda-1}=\frac{e^{\lambda-1}}{\overline{1-fz}|^{\lambda-1}}$, and substituting these values, we shall have $\dot{x}x^{\theta-1}\times\overline{e+fx^{\eta}}|^{\lambda-1}=$ $\frac{\dot{z}z^{-1}}{\eta\times\overline{1-fz}}\times\frac{e^{\frac{\theta}{\eta}}z^{\frac{\theta}{\eta}}}{\overline{1-fz}|^{\frac{\theta}{\eta}}}\times\frac{e^{\lambda-1}}{\overline{1-fz}|^{\lambda-1}}=\frac{e^{\frac{\theta}{\eta}+\lambda-1}}{\eta}\times$ $\frac{\dot{z}z^{\frac{\theta}{\eta}-1}}{\overline{1-fz}|^{\frac{\theta}{\eta}+\lambda}}=$ (If we put $\frac{\theta}{\eta}+\lambda=s$) $\frac{e^{s-1}}{\eta}\times$ $\dot{z}z^{\frac{\theta}{\eta}-1}\times\overline{1-fz}|^{-s}$, and if we compare this Fluxion with that in art. 47, we shall have $\theta=\frac{\theta}{\eta}$, $\eta=1$

$\lambda=$

$\lambda = -s$, $e=1$, $f=-f$, and the fluent will be

$$\frac{e^{s-1}}{\eta} \times \frac{\eta}{\theta} z^{\frac{\theta}{\eta}} + \frac{s.\frac{\theta}{\eta}}{\frac{\theta}{\eta}+1} Afz + \frac{\overline{s+1}.\frac{\theta}{\eta}+1}{2.\frac{\theta}{\eta}+2} Bfz + \&c.$$

$$= \frac{e^{s-1}}{\theta} \times z^{\frac{\theta}{\eta}} + \frac{s.\theta}{\theta+\eta} Afz \frac{\overline{s+1}.\theta+\eta}{2.\theta+2\eta} Bfz + \&c.$$

Q.E.I.

Example.

To find the fluent of $\frac{\dot{x}}{1+x}$ or $\dot{x} \times \overline{1+x}^{-1}$, we have $\theta = 1$, $\eta = 1$, $\lambda = 0$, $e = 1$, $f = 1$, $s = 1$, $z = \frac{x}{1+x}$ and the fluent is $z + \frac{1}{2} Az + \frac{2}{3} Bz + \frac{3}{4} Cz$ &c. $= z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \frac{1}{4} z^4$ &c.

This series is of use when the quantities e and f have the same signs, and the following when they have different ones.

50. Put $e+fx^\eta = y$, and assume $\boxed{\dot{x} x^{\theta-1} y^{\lambda-1}} = y^\lambda \times \overline{Ax^\theta + Bx^{\theta+\eta} + Cx^{\theta+2\eta}}$, &c. where A, B, C represent unknown coefficients, hereafter to be determined, by taking the Fluxion of this series we shall have $\dot{x} x^{\theta-1} y^{\lambda-1} = \theta A y^\lambda \dot{x} x^{\theta-1} + \overline{\theta+\eta} \times B y^\lambda \dot{x} x^{\theta+\eta-1} + \overline{\theta+2\eta} \times C y^\lambda \dot{x} x^{\theta+2\eta-1}$, &c. $+ \lambda A \dot{y} y^{\lambda-1} x^\theta + \lambda B \dot{y} y^{\lambda-1} x^{\theta+\eta} + \lambda C \dot{y} y^{\lambda-1} x^{\theta+2\eta}$ + &c. divide by $x^{\theta-1} y^{\lambda-1}$ then $\dot{x} = \theta A y \dot{x} + \overline{\theta+\eta} \times B y \dot{x} x^\eta + \overline{\theta+2\eta} \times C y \dot{x} x^{2\eta}$ &c. $+ \lambda A \dot{y} x + \lambda B \dot{y} x^{\eta+1} + \lambda C \dot{y} x^{2\eta+1}$ + &c. But since $y = e+fx^\eta$ we have $\dot{y} = \eta f \dot{x} x^{\eta-1}$ substitute this value of $\dot{y}$ and

$\dot{y}$ and divide by $\dot{x}$ then $1 = \theta Ay + \overline{\theta + n} \times Byx^n + \overline{\theta + 2n} \times Cyx^{2n}$ &c. $+ \lambda n f A x^n + \lambda n f B x^{2n} +$ &c. substitute the value of $y = e + fx^n$, then $1 = \theta e A + \theta f A x^n + \overline{\theta + n} \times e B x^n + \overline{\theta + n} \times f B x^{2n} + \overline{\theta + 2n} \times e C x^{2n}$, &c. $+ \lambda n f A x^n + \lambda n f B x^{2n} +$ &c. Transpose 1 and range the terms according to the dimensions of x, then

$$\left.\begin{matrix}\theta e A \\ -1\end{matrix}\right\} \left.\begin{matrix} + \theta f A \\ + \overline{\theta + n} \times e B \\ + \lambda n f A\end{matrix}\right\} x^n \left.\begin{matrix} + \overline{\theta + n} . f B \\ + \overline{\theta + 2n} . e C \\ + \lambda n f B\end{matrix}\right\} x^{2n} \text{ \&c.} = 0$$

Therefore by *Lemma* I. $\theta e A - 1 = 0$, $\overline{\theta + \lambda n} \times f A + \overline{\theta + n} \times e B = 0$, $\overline{\theta + \lambda n + n} \times f B + \overline{\theta + 2n} \times e C = 0$

And $A = \frac{1}{\theta e}$, $B = -\frac{\theta + \lambda n}{\theta + n} \times \frac{Af}{e}$, $C = -\frac{\theta + \lambda n + n}{\theta + 2n} \times \frac{Bf}{e}$, Put $\frac{\theta + n}{n} = r$ and $\frac{\theta + \lambda n}{n} = s$

then $B = -\frac{s}{r} \times \frac{Af}{e}$, $C = -\frac{s+1}{r+1} \times \frac{Bf}{e}$,

And the fluent will be equal to $\overline{e + fx^n}|^{\lambda}$ into $\frac{x^\theta}{\theta e} - \frac{s}{r} \times \frac{Af}{e} . x^{\theta + n} - \frac{s+1}{r+1} \times \frac{Bf}{e} \times x^{\theta + 2n}$ &c. Where A, B, &c. represent the coefficients of the preceeding terms, but if they represent whole terms, the fluent will be $\overline{e + fx^n}|^{\lambda} \times \frac{x^\theta}{\theta e} - \frac{s}{r} A \frac{fx^n}{e} - \frac{s+1}{r+1} B \frac{fx^n}{e} -$ &c. Q. E. I.

Example.

To find the fluent of $\dot{x}\sqrt{ax-xx}$ or $\dot{x}x^{\frac{1}{2}} \times \overline{a-x}^{\frac{1}{2}}$, we have $\theta-1=\frac{1}{2}$, $\theta=\frac{3}{2}$, $\lambda=\frac{3}{2}$, $\eta=1$, $e=a$, $f=-1$, $r=\frac{5}{2}$, $s=3$, and the fluent is $\overline{a-x}^{\frac{3}{2}} \times \frac{2x^{\frac{3}{2}}}{3a} + \frac{6}{5}\text{A}.\frac{x}{a} + \frac{8}{7}\text{B}.\frac{x}{a} + \&c. = \overline{ax-xx}^{\frac{3}{2}} \times \frac{2}{3a} + \frac{6}{5}\text{A}.\frac{x}{a} + \frac{8}{7}\text{B}.\frac{x}{a} + \frac{10}{9}\text{C}.\frac{x}{a} + \&c.$

51. Since the Quantity $\overline{e+fx^{\eta}}^{\lambda-1} = x^{\lambda\eta-\eta} \times \overline{\frac{e}{x^{\eta}}+f}^{\lambda-1}$ we have $\dot{x}x^{\theta-1} \times \overline{e+fx^{\eta}}^{\lambda-1} = \dot{x}x^{\theta+\lambda\eta-\eta-1} \times \overline{f+\frac{e}{x^{\eta}}}^{\lambda-1}$ Therefore in the last theorem, instead of θ, η, e, f, write $\theta+\lambda\eta-\eta$, $-\eta$, f, e and then $r=\frac{\theta+\eta}{\eta}$ will $=-\frac{\theta+\lambda\eta-2\eta}{\eta}$, and $s=\frac{\theta+\lambda\eta}{\eta}$ will $=-\frac{\theta-\eta}{\eta}$ and the fluent will be $\overline{f+\frac{e}{x^{\eta}}}^{\lambda} \times \frac{x^{\theta+\lambda\eta-\eta}}{f\times\overline{\theta+\lambda\eta-\eta}} - \frac{s}{r}\text{A}\frac{e}{fx^{\eta}} - \frac{s+1}{r+1}\text{B}\frac{e}{fx^{\eta}} - \&c. = \overline{e+fx^{\eta}}^{\lambda} \times \frac{x^{\theta-\eta}}{\eta f\times\overline{1-r}} - \frac{s}{r}\text{A}\frac{e}{fx^{\eta}} - \frac{s+1}{r+1}\text{B}\frac{e}{fx^{\eta}} - \&c.$ or If we put $r=\frac{\theta+\lambda\eta-2\eta}{\eta}$ and $s=\frac{\theta-\eta}{\eta}$, we shall have the fluent equal to $\overline{e+fx^{\eta}}^{\lambda} \times \frac{x^{\theta-\eta}}{\eta f\times\overline{1+r}} - \frac{s}{r}\text{A}\frac{e}{fx^{\eta}} - \frac{s-1}{r-1}\text{B}\frac{e}{fx^{\eta}} - \&c.$ This

series

series will break off when θ is any affirmative multiple of η.

Example.

To find the fluent of $\dot{x}\sqrt{a+x}$, we have $\theta=1$, $\eta=1$, $\lambda=\frac{1}{2}$, $c=a$, $f=1$, $r=\frac{1+\frac{1}{2}-2}{1}=\frac{1}{2}$, $s=0$, And the fluent is $\overline{a+x}^{\frac{3}{2}}\times\frac{2}{3}$.

Of fluents of higher orders.

52. In the same manner as the quantity $\dot{x}$ has its fluent x, the quantity x has its fluent $\boxed{x}$ which which is called the first fluent of x, and is represented thus x', this also has its fluent x'' which is the second fluent, this also has its fluent x''' which the third fluent of x, and so on.

53. These higher fluents are derived from the quantity x by taking some invariable fluxion as $\dot{w}$ which we may suppose $=1$, Then $x'=\boxed{x\dot{w}}$, $x''=\boxed{x'\dot{w}}$, $x'''=\boxed{x''\dot{w}}$, &c. In the same manner we may suppose $\dot{x}=\frac{\dot{x}}{\dot{w}}$, $\ddot{x}=\frac{\ddot{x}}{\dot{w}}$, $\dddot{x}=\frac{\dddot{x}}{\dot{w}}$, &c.

Example.

Let $x^m=y$, suppose $\dot{w}=\dot{x}=1$, then $y=x^m\dot{x}$ and $y'=\frac{x^{m+1}}{m+1}=\frac{yx}{m+1}=\frac{\dot{x}x^{m+1}}{m+1}$ therefore $y''=\frac{x^{m+2}}{m+1\,.\,m+2}=\frac{y'x}{m+2}=\frac{\dot{x}x^{m+2}}{m+1\,.\,m+2}$,

whence

whence $y''' = \frac{x^{m+3}}{m+1 \,.\, m+2 \,.\, m+3} = \frac{y''x}{m+3}$ and so on.

PROP. XI.

*To find the fluent of $\dot{z}\,v$, by the help of the higher fluxions and fluents of z and v *.*

54. Suppose $\boxed{\dot{z}v} = zv + p$, then taking the fluxions $\dot{z}v = \dot{z}v + z\dot{v} + \dot{p}$, and $\dot{p} = -z\dot{v}$, therefore $p = -\boxed{z\dot{v}} = -z'\dot{v} + q$, and taking the fluxions, $-z\dot{v} = -z\dot{v} - z'\ddot{v} + \dot{q}$, and $\dot{q} = z'\ddot{v}$, therefore $q = \boxed{z'\ddot{v}} = z''\ddot{v} + r$, taking the fluxions, $z'\ddot{v} = z'\ddot{v} + z''\dddot{v} + \dot{r}$, therefore $\dot{r} = -z''\dddot{v}$ and $r = -\boxed{z''\dddot{v}}$ therefore $\boxed{\dot{z}v} = zv - \boxed{z\dot{v}} = zv - z'\dot{v} + \boxed{z'\ddot{v}} = zv - z'\dot{v} + z''\ddot{v} - z''\dddot{v} + \text{\&c.}$ Q. E. I.

55. Or suppose $\boxed{\dot{z}v} = \dot{z}v' + p$, then taking the fluxions $\dot{z}v = \dot{z}v + \ddot{z}v' + \dot{p}$, whence $\dot{p} = -\ddot{z}v'$, And $p = -\boxed{\ddot{z}v} = -\ddot{z}v'' + q$, again taking the fluxions $-\ddot{z}v' = \ddot{z}v' - \dddot{z}v'' + \dot{q}$, therefore $\dot{q} = \dddot{z}v''$ and $q = \boxed{\dddot{z}v''} = \dddot{z}v''' + r$. Therefore $\boxed{\dot{z}v} = \dot{z}v' - \boxed{\ddot{z}v} = \dot{z}v' - \ddot{z}v'' + \boxed{\dddot{z}v'} = \dot{z}v' - \ddot{z}v'' + \dddot{z}v''' - \text{\&c.}$ Q. E. I.

* Taylor. Method. Increm. Prop. 11.

Example

Example I.

56. To find the fluent of $\dot{x}x^{\theta-1}\times\overline{e+fx^{\eta}}^{\lambda-1}$. Make $e+fx^{\eta}=y$, Then put $\dot{z}=\dot{x}x^{\theta-1}$ and $v=y^{\lambda-1}$. Also let $\dot{w}$ (art. 53.) $=\dot{y}=\eta f\dot{x}x^{\eta-1}$. Then $z=\frac{1}{\theta}x^{\theta}$, $z'=\left|\overline{z\dot{w}}\right|=\left|\frac{f\dot{x}x^{\theta+\eta-1}}{\theta}\right|=\frac{\eta fx^{\theta+\eta}}{\theta.\overline{\theta+\eta}}=\frac{\eta}{\theta+\eta}fx^{\eta}z$, $z''=\left|\overline{z'\dot{w}}\right|=\left|\frac{\eta\eta ff\dot{x}x^{\theta+2\eta-1}}{\theta.\overline{\theta+\eta}}\right|$ $=\frac{\eta\eta ffx^{\theta+2\eta}}{\theta.\overline{\theta+\eta}.\overline{\theta+2\eta}}=\frac{\eta}{\theta+2\eta}fx^{\eta}z'$. Also $v=y^{\lambda-1}$, $\dot{v}=\frac{\dot{v}}{\dot{w}}=\overline{\lambda-1}.\ y^{\lambda-2}=\frac{\overline{\lambda-1}.v}{y}$, $\ddot{v}=\overline{\lambda-1}.\ \overline{\lambda-2}.\ y^{\lambda-3}=\overline{\lambda-2}\times\frac{\dot{v}}{y}$, and substituting these values the first series $zv-z'\dot{v}+z''\ddot{v}$ &c. becomes $\frac{x^{\theta}y^{\lambda-1}}{\theta}-\frac{\eta.\overline{\lambda-1}}{\theta+\eta}A\frac{fx^{\eta}}{y}-\frac{\eta.\overline{\lambda-2}}{\theta+2\eta}B\frac{fx^{\eta}}{y}$ + &c. which is the fluent of $\dot{x}x^{\theta-1}y^{\lambda-1}$.

Suppose the fluent of $\frac{\dot{x}}{1+xx}$ was required, Then $\theta=1$, $\eta=2$, $\lambda=0$, $e=1$, $f=1$, $y=1+xx$, and the fluent is $\frac{x}{y}+\frac{2}{3}A\frac{x^2}{y}+\frac{4}{5}B\frac{x^2}{y}+$ &c.

To find the other series we have $\dot{z}=\frac{\dot{z}}{\dot{w}}=\frac{x^{\theta-\eta}}{\eta f}$, $\ddot{z}=\frac{\theta-\eta}{\eta\eta ffx^{\eta-1}}x^{\theta+\eta-1}=\frac{\theta-\eta}{\eta\eta ff}x^{\theta-2\eta}=\frac{\theta-\eta}{\eta}$ $\times\dot{z}$

$\times \frac{\dot{z}}{fx^{\eta}}$, $\dot{\dot{z}} = \frac{\theta - \eta \times \theta - 2\eta}{\eta^3 f^3 x^{\eta - 1}} x^{\theta - 2\eta - 1} = \frac{\theta - \eta . \theta - 2\eta}{\eta^3 f^3}$ $x^{\theta - 3\eta} = \frac{\theta - 2\eta}{\eta} \times \frac{\ddot{z}}{fx^{\eta}}$. Alſo $v = y^{\lambda - 1}$, $v' = \frac{1}{\lambda} y^{\lambda}$, $v'' = \frac{1}{\lambda . \lambda + 1} y^{\lambda + 1} = \frac{yv'}{\lambda + 1}$, $v''' = \frac{yv''}{\lambda + 2}$, and the ſecond ſeries $\dot{z}v' - \ddot{z}v'' + \dot{\ddot{z}}v'''$ &c. will be $\frac{x^{\theta - \eta} y^{\lambda}}{\lambda \eta f} - \frac{\theta - \eta}{\eta . \lambda + 1} A \frac{y}{fx^{\eta}} - \frac{\theta - 2\eta}{\eta . \lambda + 2} B \frac{y}{fx^{\eta}} -$ &c. which will converge the quicker the greater x^{η} is in reſpect of y or $e + fx^{\eta}$.

Example II.

57. To find the fluent of x^{m+n}. Put $\dot{z} = x^m \dot{x}$, $v = x^n$ and let $\dot{w} = \dot{x}$. Then $z = \frac{x^{m+1}}{m+1}$, $z' = \frac{x^{m+2}}{m+1 . m+2} = \frac{zx}{m+2}$, $z'' = \frac{z'x}{m+3}$. Alſo $\dot{v} = nx^{n-1} = \frac{nv}{x}$, $\ddot{v} = n . \overline{n-1} . x^{n-2} = \frac{\overline{n-1} . \dot{v}}{x}$, and the fluent $zv - z'\dot{v} + z''\ddot{v} -$ &c. $= \frac{x^{m+n+1}}{m+1} - \frac{n}{m+2} A - \frac{n-1}{m+3} B -$ &c. $=$ (by Prop. 7) $\frac{x^{m+n+1}}{m+n+1}$. Therefore the ſum of the infinite ſeries $\frac{1}{m+1} - \frac{n}{m+3} A - \frac{n-1}{m+2} B -$ &c. $= \frac{1}{m+n+1}$:

Example

Example III.

58. Suppose $\dot{A} = \frac{\dot{x}}{1+x^2}$, to find the fluent of $\dot{x}x^{m-1}A$. Where m is an even number. Let $\dot{w} = \dot{A}$, put $\dot{z} = x^{m-1}\dot{x}$, $v = A$, then $\dot{v} = 1$, $\ddot{v} = 0$, $z = \frac{1}{m}x^m$, $z' =$ the fluent of $\frac{1}{m}x^m\dot{A} =$ fluent of $\frac{1}{m} \times \frac{x^m\dot{x}}{1+x^2}$. But $\frac{x^m\dot{x}}{m \times \overline{1+x^2}}$ or $-\frac{x^m\dot{x}}{m \times \overline{x^2+1}} = \frac{\dot{x}x^{m-2}}{m} - \frac{\dot{x}x^{m-4}}{m} \ldots - \frac{\dot{x}}{m \cdot \overline{x^2+1}} = \frac{\dot{x}x^{m-2}}{m} - \frac{\dot{x}x^{m-4}}{m} \ldots - \frac{\dot{A}}{m}$ whose fluent $= z' = \frac{x^{m-1}}{m.\overline{m-1}} - \frac{x^{m-3}}{m.\overline{m-3}} \ldots - \frac{A}{m}$, and $zv - z'\dot{v} = \frac{1}{m}x^mA - \frac{x^{m-1}}{m.\overline{m-1}} + \frac{x^{m-3}}{m.\overline{m-3}} \ldots + \frac{A}{m} =$ The fluent of $\dot{x}x^{m-1}\dot{A}$.

59. If we are to find the fluent of a quantity composed of more flowing quantities than one, if this can be reduced to any of these forms. $\dot{x}y + x\dot{y}$, $\dot{x}yz + x\dot{y}z + xy\dot{z}$, we may find the fluent by Prop. 5 and its Corollaries, thus the quantity $m\dot{x}x^{m-1}y^n + nx^m\dot{y}y^{n-1}$ or $x^{m-1}y^{n-1} \times \overline{m\dot{x}y + nx\dot{y}}$ by substituting z for x^m and v for y^n is reduced to $\dot{z}v + z\dot{v}$ whose fluent is $zv + C = x^my^n + C$.

60. If the relation of the fluxions is $\frac{\dot{t}}{t} + \frac{\dot{v}}{v} + \frac{\dot{x}}{x}$ &c. $= \frac{\dot{T}}{T} + \frac{\dot{V}}{V} + \frac{\dot{X}}{X} +$ &c. Then by art.

21,

21. tvx &c. is to TVX in a constant ratio, or if C is a constant quantity $Ctvx = TVX$.

If $\frac{m\dot{z}}{z} = \frac{\dot{x}}{x}$, put $z^m = y$ then $\dot{y} = \frac{m\dot{z}z^m}{z} = \frac{m\dot{z}y}{z}$ and $\frac{m\dot{z}}{z} = \frac{\dot{y}}{y} = \frac{\dot{x}}{x}$ Therefore $y = Cx$ or $z^m = Cx$,

If $\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y} = \frac{\dot{z}}{z}$ then $x^m y^n = Cz$.

If $m\dot{x}z^n + n\dot{z}xz^{n-1} = \dot{x}x^p$ multiply by x^{m-1}, then $m\dot{x}x^{m-1}z^n + n\dot{z}x^m z^{n-1} = \dot{x}x^{p+m-1}$ and taking the fluents by art. 59. $x^m z^n = \frac{x^{p+m}}{p+m} + C$.

And if C should be $= 0$, $z^n = \frac{x^p}{p+m}$.

Prop. XII.

To find the root y of an adfected literal or fluxional equation, involving only two flowing quantities x and y where x flows uniformly and its fluxion $\dot{x} = 1$.

61. Suppose it was required to find the root y of the equation $y^3 - a^2y + x^2y - a^2x = 0$, when x is very small. Or to express y by a series converging the swifter the less x is. Since x is nearly $= 0$, we shall have $y^3 - a^2y$ nearly $= 0$ and $y = a$ nearly. For the same reason $a^2y + a^2x$ is nearly $= 0$, and $y = -x$ nearly. Suppose $y = Ax^m$ nearly, then substituting this expression for y in the given equation, we have $A^3x^{3m} - a^2Ax^m + Ax^{m+2} - a^2x = 0$. Now since we are to assume only such terms to determine y or Ax^m, as will become

D much

much greater than the other terms, the index of the assumed terms must be less than any of the other indices $3m$, $m+2$, &c.

62. To determine this index, draw the rectangle A B C D (fig. 2.) and divide it into the equal squares 0, 1, m, &c. And taking any square in the line A D for 0, in the lateral squares on the right hand, write m, $2m$, $3m$. &c. and on the row next above that 1, $m+1$, $2m+1$, &c. and on the next row 2, $m+2$, $2m+2$, &c. and so on. From the description of this parallelogram, it is evident that selecting any two squares as 2 and m, and making the numbers in them equal and thence determining m, and supposing a ruler E F to touch their angular points, then all those squares which are above the ruler will be greater, and those below will be less than those first selected and whose angular points touch the ruler. And this property will hold good although the numbers of some squares be wanting.

63. Hence is deduced a method for finding the index m, for having substituted Ax^m for y, And placed the indices 1, m, $m+2$, &c. in their proper places in the parallelogram, make a ruler to revolve about some external square, till it meets some other external square, either above it, or in the same row, then the numbers in those squares being made equal will determine m, and the respective terms with those indices being made equal to 0, will give A. Thus in the above example (art. 61.) suppose a ruler to revolve about $3m$, till it meets m, then making

	$m+2$		
1			
	m		$3m$

$3m=m$

$3m=m$, you will find $m=0$, also making $A^3x^{3m}-a^2Ax^m=0$, or $A^3-a^2A=0$, we shall have $A=\pm a$ and 0, therefore Ax^0 or $\pm a$, is the first term of a series expressing y, when x is very small.

In the same manner if a ruler revolve about m till it meets 1, then make $m=1$, And $-a^2Ax^m-a^2x=0$, or $A=-1$, and Ax or $-x$ is the first term of a series for y when x is very small.

64. If we suppose the ruler to move parallel to itself, all the numbers which at the same time touch the ruler will be equal, and those which successively touch it, will be contained in an arithmetical progression. Thus in the above example, the ruler first arrives at m and $3m$ or 0, next at 1, and last at $m+2$ or 2. Now 0, 1, 2, are contained in an arithmetical progression whose common difference is 1.

65. Having substituted this value of m in the equation it becomes $\left.\begin{matrix} A^3 \\ -a^2A \end{matrix}\right\} - a^2x + Ax^2 = 0.$

Now suppose the series expressing y be of this form, $Ax^m + Bx^{m+n} + Cx^{m+2n}$, &c. where the common difference of the indices is n, it is evident from Lemma 2. that in any power of y, the common difference of the indices will be the same. Therefore to make these indices coincide with those of the other terms of the equation, so that they may be compared together to determine the coefficients A, B, C, &c. the common difference of the indices, n must be the same with the common difference of the arithmetical progression in which the indices m, 1, $m+2$, &c. are contained.

66. Having now the form of the series expressing y, substitute this series for y in the equation, and determine the coefficients by Lemma I. Thus in the above example, the form of one of the series is $y=A+Bx+Cx^2+Dx^3+$ &c. The form of the other series where $-x$ is the first term of y is found to be $Ax+Bx^3+Cx^5+$ &c. because the indexes m or 1, and $3m=m+2$ or 3 is contained in the arithmetical progression $1, 3, 5,$ &c.

The Coefficients of the first series are thus found.

$$\left.\begin{array}{rllll} y^3 = & A^3+3A^2Bx & +3AB^2x^2 & + B^3x^3 \\ & * \quad\quad * & +3A^2C & +6ABC \\ & * \quad\quad * & \quad * & +3A^2D \\ -a^2y = & -a^2A-a^2B & -a^2C & -a^2D \\ +x^2y = & * \quad\quad * & +A & + B \\ -a^2x = & * \quad -a^2 & \quad * & \quad * \end{array}\right\}\text{\&c.}=0$$

Therefore by Lemma I. $A^3-a^2A=0$, and $A=+a, -a, 0.$ $3A^2B-a^2B-a^2=0$, $B=\frac{a^2}{3A^2-a^2}=+\frac{1}{2}, \frac{1}{2}, -1.$

$3AB^2+3A^2C-a^2C+A=0$, $C=-A\times\frac{1+3B^2}{3A^2-a^2}, =\frac{-7}{8a}, \frac{+7}{8a}, 0.$ $B^3+6ABC+3A^2D-a^2D+B=0$, $D=-B\times\frac{1+6AC+B^2}{3A^2-a^2}=\frac{1}{a^2}, \frac{1}{a^2}, \frac{-2}{a^2}$, therefore the three series for y are

$y=$

$$y = \quad a + \tfrac{1}{2}x - \frac{7x^2}{8a} + \frac{x^3}{a^2}, \text{ \&c.}$$

$$y = -a + \tfrac{1}{2}x + \frac{7x^2}{8a} + \frac{x^3}{a^2}, \text{ \&c.}$$

$$y = * \; - x \; * \; - \frac{2x^3}{a^2}, \text{ \&c.}$$

The second case is thus resolved,

$$\left.\begin{array}{lllll} -a^2y = & -a^2Ax & -a^2Bx^3 & -a^2Cx^5 \\ -a^2x = & -a^2x & * & * \\ +y^3 = & * & +A^3 & +3A^2B \\ +x^2y = & * & +A & +B \end{array}\right\} \text{\&c.} = 0$$

Therefore $-a^2A - a^2 = 0$, And $A = -1$.
$-a^2B + A^3 + A = 0$ and $B = A \times \frac{1+A^2}{a^2} = \frac{-2}{a^2}$. $-a^2C + 3A^2B + B = 0$ and $C = B \times \frac{1+3A^2}{a^2} = \frac{-8}{a^4}$. Therefore $y = -x - \frac{2x^3}{a^2} - \frac{8x^5}{a^4} -$ &c. which is the same as the third series in the last case.

67. When the quantity x is supposed very large, then the index of the terms assumed to determine y or Ax^m must be greater than any of the other indices: This is determined by making the ruler revolve about some external square, till it touch another external square, either below it or in the same row, and making the numbers in them equal you may determine m and A. And by making the ruler move parallel to itself, you will find the form of the series, in the same manner as before.

Example I.

In the equation of art. 61. making the ruler revolve about $m+2$, till it meets 1, then making $m+2=1$, and $A-a^2=0$, you will find $m=-1$, $A=a^2$, and the ruler will successively arrive at 1, m or -1, and $3m$ or -3, therefore the form of the series is $y=Ax^{-1}+Bx^{-3}+Cx^{-5}+Dx^{-7}$, &c. whose coefficients are thus found;

$$\left.\begin{array}{rcccc} +yx^2 = & Ax & +Bx^{-1} & +Cx^{-3} & +Dx^{-5} \\ -a^2x = & -a^2 & * & * & * \\ -a^2y = & * & -a^2A & -a^2B & -a^2C \\ +y^3 = & * & * & +A^3 & +3A^2B \end{array}\right\} \text{\&c.}=0.$$

Therefore $A-a^2=0$, and $A=a^2$, $B-a^2A=0$ and $B=a^2A=a^4$, $C-a^2B+A^3=0$, and $C=a^2B-A^3=0$, $D-a^2C+3A^2B=0$, and $D=a^2C-3A^2B=-3a^8$. Therefore $y=\frac{a^2}{x}+\frac{a^4}{x^3} * -\frac{3a}{x^7}$,

Here if we make the ruler to revolve about $m+2$ till it meets $3m$, then will $m+2=3m$, and $m=1$, but $A^3+A=0$, whose root is 0, which will give the same series as before, the other two roots are impossible, which shews that no descending series can be found whose first term is Ax.

Example II.

68. To find the root y of the fluxional equation $\dot{y}+xy-x-1=0$. Where $\dot{x}=1$. Suppose as before $y=Ax^m$ nearly, then $\dot{y}=mAx^{m-1}$, and substituting these values $mAx^{m-1}+Ax^{m+1}$

—

2	$m+2$
1	
	m

$-x-1=0$, or multiplying by x, $mAx^m+Ax^{m+2}-x^2-x=0$. And placing these indices in the Parallelogram, it appears that we may have both an ascending and descending series for y. When x is very small suppose the ruler to be applied to m and 1 then $m=1$, and the ruler will successively arrive at m or 1, 2, and $m+2$ or 3. Therefore the form of the series to be assumed for y is $Ax+Bx^2+Cx^3+Dx^4+$ &c. Whence,

$$\left.\begin{array}{llllll} +\dot{y}= & A & +2Bx & +3Cx^2 & +4Dx^3 \\ +xy= & * & * & +A & +B \\ -x= & * & -1 & * & * \\ -1= & -1 & * & * & * \end{array}\right\}\text{\&c.}=0.$$

Therefore $A-1=0$, and $A=1$, $2B-1=0$, and $B=\frac{1}{2}$, $3C+A=0$, and $C=-\frac{1}{3}A=-\frac{1}{3}$, $4D+B=0$ and $D=-\frac{1}{4}B=-\frac{1}{8}$, therefore $y=x+\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{8}x^4$, &c. When x is supposed very large, let the ruler be applied to $m+2$ and 2, then $m+2=2$ and $m=0$, and the ruler will successively arrive at 2, 1, and m or 0, therefore assume $y=A+Bx^{-1}+Cx^{-2}+Dx^{-3}$, &c.

$$\left.\begin{array}{llllll} +xy= & Ax & +B & +Cx^{-1} & +Dx^{-2} \\ +\dot{y}= & * & * & * & -B \\ -x= & -1 & * & * & * \\ -1= & * & -1 & * & * \end{array}\right\}\text{\&c.}=0.$$

Therefore $A-1=0$, and $A=1$, $B-1=0$ and $B=1$, $C=0$, $D-B=0$ and $D=B=1$, therefore

$y=1+\frac{1}{x}*+\frac{1}{x^3}$, &c.

Example III.

69. To find the root of the fluxional equation $\dot{y}+x\ddot{y}-x\dot{y}-x^2\ddot{y}-1=0$; Put $y=Ax^m$, then $\dot{y}=mAx^{m-1}$, $\ddot{y}=\overline{m^2-m}\times Ax^{m-2}$, and the equation will become $mAx^{m-1}+\overline{m^2-m}\times Ax^{m-1}-mAx^m-\overline{m^2-m}\times Ax^m-1=m^2Ax^{m-1}-m^2Ax^m-1=0$, or multiplying by x, $m^2Ax^m-m^2Ax^{m+1}-x=0$.

1	$\frac{m+1}{m}$

When x is very ſmall $m=1$, and the form of the ſeries is $y=Ax+Bx^2+Cx^3+Dx^4$, &c.

$$\left.\begin{array}{rcccc} \dot{y}= & A & +2Bx & +3Cx^2 & +4Dx^3 \\ +x\ddot{y}= & * & +2B & +6C & +12D \\ -x\dot{y}= & * & -A & -2B & -3C \\ -x\ddot{y}= & * & * & -2B & -6C \\ -1= & -1 & * & * & * \end{array}\right\}\text{\&c.}=0.$$

Therefore $A-1=0$ and $A=1$. $4B-A=0$ and $B=\frac{1}{4}A=\frac{1}{4}$, $9C-4B=0$ and $C=\frac{4}{9}B=\frac{1}{9}$, $16D-9C=0$ and $D=\frac{9}{16}C=\frac{1}{16}$. Therefore $y=x+\frac{1}{4}x^2+\frac{1}{9}x^3+\frac{1}{16}x^4$, &c. When x is very large $m+1=1$ or $m=0$, and $m^2A+1=0$. Whence $A=\frac{-1}{m^2}=\frac{1}{0\times0}$ which is infinite, therefore when x is very great, y is infinite.

70. In the ſame manner we may find the root when $\dddot{y}$, $\ddddot{y}$, or any other of its fluxions, or any power of a fluxion is found in the equation, by ſubſtituting Ax^m for y, mAx^{m-1} for $\dot{y}$, $m.\overline{m-1}.Ax^{m-2}$ for $\ddot{y}$, $m.\overline{m-1}.\overline{m-2}$.

A

$Ax^{m}{}_{-3}$ for $\dot{\ddot{y}}$, and A^2x^{2m-2} for $\dot{y}^2, m^2 . m-1 . Ax^{2m-3}$ for $\dot{y}\ddot{y}$, &c. and then finding the form of the series and the coefficients as before.

71. Since when x is very small, those terms in which x is of least dimensions, and when x is very large, those in which it is of most dimensions, are to vanish, we may make the coefficients $m, m . \overline{m-1}$ &c. of those terms in which x in each case is of least and greatest dimensions to vanish or $=0$, and thence determine m*, but A will not be determined by the comparison of terms, and is only to be known from the conditions of the flowing quantity y.

Example.

Suppose the equation be $y+\ddot{y}=0$. Suppose $y=Ax^{m}$, then $Ax^{m}+m . \overline{m-1} . Ax^{m-2}=0$. Now when x is very small the quantity $m . \overline{m-1} . Ax^{m-2}$ where x is of least dimensions is to vanish, therefore put $m^2-m=0$ and $m=0$ or 1. And as the difference of the two indices m and $m-2$ is 2, the form of the two serieses will be $y=A+Bx^2+Cx^4+Dx^5$ &c. and $y=Ax+Bx^3+Cx^5+Dx^7$ &c. whose coefficients are thus determined.

$$\left.\begin{array}{r} y=\ A+Bx^2+Cx^4 \\ +\ddot{y}=2B+12C+30D \end{array}\right\}\text{\&c.}=0.$$

Therefore $2B+A=0$ and $B=\frac{-1}{2}A$, $12C+$

* Stirling. Summat. Serier. Prop. XV. Schol.

$B = 0$ and $C = \frac{1}{12} B = + \frac{1}{24}$ A, $30 D +$ $C = 0$ and $D = - \frac{1}{30} C = - \frac{1}{720}$ A, therefore $y = A - \frac{1}{2} Ax^2 + \frac{1}{24} A x^4 - \frac{1}{720} A x^6 +$ &c.

The other series is thus determined

$$\left.\begin{array}{r} y = Ax + Bx^3 + Cx^5 \\ + \ddot{y} = 6B + 20C + 42D \end{array}\right\} \text{\&c.} = 0.$$

Therefore $A + 6B = 0$ and $B = - \frac{1}{6}$ A, $B + 20$ $C = 0$ and $C = - \frac{1}{20} B = + \frac{1}{120} A$. $C + 42D = 0$ & $D = - \frac{1}{42} C = - \frac{1}{5040}$ A. And $y = Ax -$ $\frac{1}{6} Ax^3 + \frac{1}{120} Ax^5 - \frac{1}{5040} Ax^7$ &c.

When x is very large, the term Ax^m must vanish and $A = 0$. Therefore y must $= 0$.

72. When we have given an equation of this form $x = ay + by^2 + cy^3 + dy^4 +$ &c. to find y. This is called the reversion of a series. Substitute Ax^m for y, then $0 = - x + aAx^m + bA^2x^{2m} +$ cA^3x^{3m} &c. and placing these indices in the parallelogram it appears that $m = 1$, and the form of the series for y is $Ax +$ $Bx^2 + Cx^3 + Dx^4$, &c.

1			
	m	$2m$	&c.

Which being substituted in the equation we have

ay

$$\left.\begin{array}{rcccccc}
ay & = & aAx & + aBx^2 & + aCx^3 & + aDx^4 \\
+ by^2 & = & * & + bA^2 & + 2bAB & + 2bAC \\
 & & * & * & * & + bB^2 \\
+ cy^3 & = & * & * & + cA^3 & + 3cA^2B \\
+ dy^4 & = & * & * & * & + dA^4 \\
- x & = & -1 & * & * & *
\end{array}\right\} \&c. = 0.$$

Therefore $aA - 1 = 0$ and $A = \frac{1}{a}$, $aB + bA^2 = 0$ and $B = -\frac{bA^2}{a} = -\frac{b}{a^3}$, $aC + 2bAB + cA^3 = 0$ and $C = -\frac{cA^3 + 2bAB}{a} = \frac{2b^2 - ac}{a^5}$, $aD + 2bAC + bB^2 + 3cA^2B + dA^4 = 0$, and

$$D = -\frac{2bAC + bB^2 + 3cA^2B + dA^4}{a} = + \frac{2abc - 4b^3 - b^3 + 3abc - a^2d}{a^7} = \frac{5abc - 5b^3 + a^2d}{a^7},$$

and therefore

$$y = \frac{x}{a} - \frac{bx^2}{a^3} + \frac{2bb - ac}{a^5} x^3 + \frac{5abc - 5b^3 - a^2d}{a^7} x^4 + \&c.$$

Example.

Let $x = y + y^2 + y^3 + y_4$ &c. Then $a = 1$, $b = 1$, $c = 1$, and $y = x - x^2 + x^3 - x^4 +$ &c.

73. If the series be $x = ay + by^3 + cy^5 + dy^7 +$ &c. Then the form of the series is $y = Ax + Bx^3 + Cx^5 + Dx^7$ &c.

$$\left.\begin{array}{llllll} ay= & aAx & +aBx^3 & +aCx^5 & +aDx^7 \\ +by^3= & * & +bA^3 & +3bA^2B & +3bAB^2 \\ & * & * & * & +3bA^2C \\ +cy^5= & * & * & +cA^5 & +5cA^4B \\ +dy^7= & * & * & * & +dA^7 \\ -x= & -1 & * & * & * \end{array}\right\} \&c. = 0.$$

Therefore $aA-1=0$ and $A=\frac{1}{a}$, $Ba+bA^3=0$ and $B=-\frac{bA^3}{a}=-\frac{b}{a^4}$, $aC+3bA^2B+cA_5=0$ and $C=-\frac{3bA^2B+cA^5}{a}=-\frac{3b^2-ac}{a^7}$, $aD+3bAB^2+3bA^2C+5cA^4B+dA^7=0$ and $D=-\frac{3bAB^2+3bA^2C+5cA^4B+dA^7}{a}=-\frac{3b^3-9b^3+3abc+5abc-a^2d}{a^{10}}=\frac{8abc-12b^3-a^2d}{a^{10}}$ Therefore $y=\frac{x}{a}-\frac{bx^3}{a^4}+\frac{3bb-ac}{a^7}x^5+\frac{8abc-12b^3-a^2d}{a^{10}}x^7+$ &c. This and the last are Sir Isaac Newton's two theorems for reversing of series *.

74. If the equation be $fx+gx^2+hx^3+kx_4+$ &c. $=y+by^2+cy^3+dy^4$ &c. The form of the series for y will be $Ax+Bx^2+Cx^3+Dx_4+$ &c.

* Letter to Mr. Oldenburg. 24 Oct. 1676.

$y=$

$$\left.\begin{array}{rcccccl}
y & = & Ax & +Bx^2 & +Cx^3 & +Dx^4 \\
+by^2 & = & * & +bA^2 & +2bAB & +2bAC \\
 & & * & * & * & +bB^2 \\
+cy^3 & = & * & * & +cA^3 & +3cA^2B \\
+dy^4 & = & * & * & * & +dA^4 \\
\&c. & & & & & \&c. \\
-fx & = & -f & * & * & * \\
-gx^2 & = & * & -g & * & * \\
-hx^3 & = & * & * & -h & * \\
-kx^4 & = & * & * & * & -k
\end{array}\right\} \&c. = 0.$$

Therefore $A-f=0$ and $A=f$, $B+bA^2-g=0$ and $B=g-bA^2$, $C+2bAB+cA^3-h=0$ and $C=h-2bAB-cA^3$, $D+2bAC+bB^2+3cA^2B+dA^4-k=0$ and $D=k-2bAC-3cA^2B-dA^4-bB^2$ and A, B, C, D, being thus found we shall have $y=Ax+Bx^2+Cx^3+Dx^4+$ &c.

Prop. XIII *.

75. *Having an equation involving two flowing quantities* x *and* y *of which* y *flows uniformly, to transform it so that* x *shall flow uniformly.*

Let A, B, C, &c. be any quantities composed out of the powers of x and given quantities, and let $\dot{y}=A\dot{x}$, $\dot{A}=B\dot{x}$, $\dot{B}=C\dot{x}$, &c. then taking the fluxions $\ddot{y}=\dot{A}\dot{x}+A\ddot{x}=B\dot{x}\dot{x}+A\ddot{x}$, $\dddot{y}=\dot{B}\dot{x}\dot{x}+2B\dot{x}\ddot{x}+\dot{A}\ddot{x}+A\dddot{x}=C\dot{x}^3+3B\dot{x}\ddot{x}+A\dddot{x}$, Therefore $A=\frac{\dot{y}}{\dot{x}}$, $B=\frac{\ddot{y}-A\ddot{x}}{\dot{x}\dot{x}}$ and $C=$

* Taylor Meth. Incr. Prop. 3.

$\dddot{y}-$

$\frac{\dddot{y}-3B\dot{x}\ddot{x}-A\dddot{x}}{\dot{x}^3}$ Now if y flows uniformly, we have $A=\frac{\dot{y}}{\dot{x}}$, $B=\frac{-A\ddot{x}}{\dot{x}^2}$, $C=\frac{-3B\dot{x}\ddot{x}-A\dddot{x}}{\dot{x}^3}$. But if x flows uniformly we have $A=\frac{\dot{y}}{\dot{x}}$ $B=\frac{\ddot{y}}{\dot{x}^2}$, $C=\frac{\dddot{y}}{\dot{x}^3}$ Therefore $-A\ddot{x}$ is transformed into $\ddot{y}$ and $\ddot{x}$ into $\frac{-\ddot{y}}{A}=\frac{-\dot{x}\ddot{y}}{\dot{y}}$, Also $-3B\dot{x}\ddot{x}-A\dddot{x}$ is transformed into $\dddot{y}$ and $\dddot{x}$ into $\frac{-3B\dot{x}\ddot{x}-\dddot{y}}{A}$ but $B=\frac{\ddot{y}}{\dot{x}^2}$ and $\ddot{x}=-\frac{\dot{x}\ddot{y}}{\dot{y}}$, therefore $B\dot{x}\ddot{x}=\frac{-\ddot{y}^2}{\dot{y}}$ and $\dddot{x}=\frac{-\dddot{y}\dot{x}\dot{y}+3\ddot{y}^2\dot{x}}{\dot{y}\dot{y}}$, therefore substitute $\frac{-\dot{x}\ddot{y}}{\dot{y}}$ for $\ddot{x}$ and $\frac{3\ddot{y}^2\dot{x}-\dot{x}\dot{y}\dddot{y}}{\dot{y}\dot{y}}$ for $\dddot{x}$ in the equation and the problem will be solved. Q. E. F.

SECT.

Sect. III.

Of Logarithms and Exponential Quantities.

Definitions.

76. The measure of a ratio is a quantity of any kind which is proportional to the quantity or magnitude of that ratio.

The Logarithm of any number is the measure of the ratio of that number to 1. Thus the logarithm of A is the measure of the ratio of A : 1.

77. The Logarithms of two numbers added together are equal to the Logarithm of their product; because the ratio of A : 1 being added to the ratio of B : 1, the sum is the ratio of AB : 1. The logarithm of one number being substracted from the logarithm of another number, the remainder is the logarithm of the quotient arising from the division of the last number by the first; because the ratio A : 1 subtracted from the ratio B : 1 leaves the ratio $\frac{B}{A} : 1$.

The logarithm of any number A being multiplied by any number as m, the product is the logarithm of the power A^m, because the ratio $A^m : 1$ is m times the ratio of A : 1.

If any number of quantities are in geometrical progression, their logarithms will be arithmetical progression. For let A, B, C, D, &c. be in geometrical proportion, and a, b, c, d, their logarithms, then will $\frac{A}{B} = \frac{B}{C} = \frac{C}{D}$ and therefore $a - b = b - c = c - d$.

Lemma

Lemma III.

78. *Let* $a+x$ *and* a *be two quantities whose difference* x *is indefinitely small in respect of* a, *I say the measure of the ratio* $a+x : a$ *is proportional to* $\frac{x}{a}$ *or that the ratio* $1+\frac{mx}{a} : 1$ *or* $a+mx : a$ *is* m *times the ratio of* $1+\frac{x}{a} : 1$ *or* $a+x : a$.

For m times the ratio $1+\frac{x}{a} : 1$ is the ratio of $\overline{1+\frac{x}{a}}|^m : 1$ or by Lemma 2. the ratio of $1+\frac{mx}{a}+m\times\frac{m-1}{2}\times\frac{x^2}{a^2}+$ &c. : 1. Now suppose the quantity $\frac{x}{a}$ diminished *ad infinitum*, and the last ratio of $\overline{1+\frac{x}{a}}|^m : 1$ will become the ratio of $1+\frac{mx}{a} : 1$. Q. E. D.

PROP. XIV.

To find the measure of any ratio *.

79. Suppose it was required to find the measure of the ratio between $a+b$ and a, suppose the difference b to be divided into an infinite number of infinitely small equal parts x, and the ratio $a+b : a$ will be equal to the sum of all the ratios $a+x : a$, $a+2x : a+x$, $a+3x : a+2x$; but by Lem-

* Cotes Logom. Prop. 1.

ma

ma 3, the measure of the ratio $a+x:a$ is as $\frac{x}{a}$, the measure of $a+2x:a+x$ is as $\frac{x}{a+x}$, and in general the measure of the ratio $a+\overline{m+1}\times x:$ $a+mx$ is as $\frac{x}{a+mx}$.

Call the quantity mx, y, then the increment of y will be x, and the measure of the ratio $a+\overline{m+1}\times x$ $:a+mx$ that is the increment of the ratio of $a+b:$ a will be as $\frac{x}{a+y}$ or since the fluxions are as the increments, The fluxion of the measure of the ratio $a+b:a$ will be as $\frac{\dot{y}}{a+y}$; or taking a proper quantity as M the measure of the ratio $a+b:a$ will be equal to the fluent of $\frac{M\dot{y}}{a+y}$ when $y=b$. Q. E. I.

Corol. 1. The measure of the ratio $a:y$ will be equal to the fluent of $M\times\frac{\dot{y}}{y}$.

Definition.

The quantity M is called the *Modulus* of the system of measures or logarithms.

Corol. 2. If the ratio between $a+y$ and a be given then the fluent of $\frac{\dot{y}}{a+y}$ will be given, and the fluent of $\frac{M\dot{y}}{a+y}$ will be as M. Therefore the modulus of

any system is as the measure of any given ratio, and is therefore, in every system, the measure of a certain given ratio, which ratio is called by Mr. COTES the modular ratio.

80. The fluent of $\frac{\dot{y}}{a+y}$ by art. 46. is $\frac{y}{a} - \frac{y^2}{2a^2} + \frac{y^3}{3a^3} - \frac{y^4}{4a^4} + \&c.$ and therefore the measure of the ratio $a+y : a$ is equal to $M \times \overline{\frac{y}{a} - \frac{y^2}{2a^2} + \frac{y^3}{3a^3} - \&c.}$ And if $a=1$ the logarithm of $1+y$ will be $M \times \overline{y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \&c.}$ which series will converge the quicker the less y is in respect of 1.

If we put $\frac{y}{1+y} = z$, and comparing the above fluxion with that in art. 49 we shall have the fluent $= M \times \overline{z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \&c.}$

81. If the logarithm of $\frac{a+x}{a-x}$ be required, let $y = \frac{a+x}{a-x}$, then $\dot{y} = \frac{2a\dot{x}}{\overline{a-x}|^2}$ and $\frac{\dot{y}}{y} = \frac{2a\dot{x}}{a^2-x^2}$ whose fluent by art. 47 is $\frac{2x}{a} + \frac{1}{3} A \frac{x^2}{a^2} + \frac{3}{5} B \frac{x^2}{a^2} + \frac{5}{7} C \frac{x^2}{a^2} + \&c.$ and therefore by Cor. I. Prop. XIV. The logarithm of $\frac{a+x}{a-x}$ is $2M \times \overline{\frac{x}{a} + \frac{1}{3} A \frac{x^2}{a^2} + \frac{3}{5} B \frac{x^2}{a^2} + \frac{5}{7} C \frac{x^2}{a^2}} = 2M \times \overline{\frac{x}{a} + \frac{x^3}{3a^3} + \frac{x^5}{5a^5} + \&c.}$

82. If we are to find the logarithm of $a+x$ which shall converge the swifter the greater x is in respect of

of a, the fluxion of the logarithm is $\frac{M\dot{x}}{x+a} = \frac{M\dot{x}}{x} - \frac{Ma\dot{x}}{x^2} + \frac{Ma^2\dot{x}}{x^3} - \frac{Ma^3\dot{x}}{x^4}$ &c. whose fluent is (because the fluent of $\frac{M\dot{x}}{x}$ is log. x) $=$ log. $x + M \times \overline{\frac{a}{x} - \frac{a^2}{2x^2} + \frac{a^3}{3x^3}}$, &c.

83. To find the logarithm of the infinite series $1 + Ax + Bx^2 + Cx^3 + Dx^4$ &c. The fluxion of the logarithm is $M\dot{x} \times$

$$\frac{A + 2Bx + 3Cx^2 + 4Dx^3 + \&c.}{1 + Ax + Bx^2 + Cx^3 \; \&c.} = M\dot{x} \times$$

$$A + \overline{2B - A^2} \times x + \overline{3C - 3AB + A^3} \times x^2$$

$$+ \overline{4D - 4AC + 4A^2B - A^4 - 2B^2} \times x^3 + \&c.$$

Whose fluent is $M \times Ax + \frac{2B - A^2}{2} \times x^2 +$

$$\frac{3C - 3AB + A^3}{3} \times x^3 +$$

$$\frac{4D - 4AC + 4A^2B - A^4 - 2B^2}{4} \times x^4 + \&c.$$

Prop. XV.

Having given a logarithm, to find the number to which it belongs.

84. Call the logarithm ϱ, the number $1 + y$, then by art. 80. $\varrho = M \times \overline{y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4}$, &c. And $\frac{\varrho}{M} = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$ &c. This series re-

reversed by art. 72. gives $y = \frac{\varrho}{M} + \frac{\varrho^2}{2M^2} + \frac{\varrho^3}{6M^3} + \frac{\varrho^4}{24M^4}$, &c. and the number $1+y$ is $1 + \frac{\varrho}{M} + \frac{\varrho^2}{2M^2} + \frac{\varrho^3}{6M^3} + \frac{\varrho^4}{24M^4} +$ &c. Q. E. I.

Corol. Here if we suppose $\varrho = M$ the modular ratio (Corol. 2. Prop. XIV.) is that of $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$ + &c. to 1 or $1 + 1 + \frac{1}{2} + \frac{1}{2\times3} + \frac{1}{2\times3\times4}$, &c. to 1.

85. Let the number be $\frac{1+x}{1-x}$ then by art. 81. $\varrho = 2M \times \overline{x + \frac{1}{3}Ax^2 + \frac{3}{5}Bx^2 + \frac{5}{7}Cx^2 + \&c.}$ and $\frac{\varrho}{2M} = x + \frac{1}{3}Ax^2 + \frac{3}{5}Bx^2 +$ &c. $= x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7$ &c. which reversed by art. 73 gives $x = \frac{\varrho}{2M} - \frac{\varrho^3}{3\times8M^3} + \frac{\varrho^5}{15\times16M^5} - \frac{17\varrho^7}{315\times128M^7} +$ &c. and x being found we have the number $\frac{1+x}{1-x}$.

86. To find the number whose logarithm is the infinite series $M \times \overline{ax + bx^2 + cx^3 + \&c.}$ Suppose the number to be $1 + Ax + Bx^2 + Cx^3$, &c. and comparing the logarithm of this series in art. 83 with the given logarithm, we have $A = a$, $2B - A^2 = 2b$ and $B = \frac{2b + a^2}{2}$, $3C - 3AB + A^3 = 3c$ and

$C =$

$C = \frac{3c + 3AB - A^3}{3} = \frac{3c + 3ab + \frac{3}{2}a^3 - a^3}{3} =$

$\frac{6c + 6ab + a^3}{6}$ and the number is $1 + ax + \frac{2b + a^2}{2}$

$\times x^2 + \frac{6c + 6ab + a^3}{6} \times x^3 +$ &c.

Prop. XVI.

To find the fluents of quantities which involve both a flowing quantity and its logarithm as $\dot{x}x^{m-1}X^n$ *where* X *represents the logarithm of* x.

87. Let $\dot{z} = \dot{x}x^{m-1}$ and $v = X^n$ and let $\dot{w}$ (art. 53.) $= \dot{X} = \frac{\dot{x}}{x}$ then $z = \frac{1}{m}x^m$, $z' = \left|\frac{1}{m}x^{m-1}\dot{x}\right| = \frac{1}{m^2}x^m$,

$z'' = \frac{1}{m^3}x^m$, also $\dot{v} = nX^{n-1}$, $\ddot{v} = n \times \overline{n-1} \times X^{n-2}$, and the fluent by Prop. XI. is $zv - z'\dot{v} + z''\ddot{v} -$ &c. $= \frac{1}{m}x^m X^n - \frac{n}{m^2}x^m X^{n-1} + \frac{n \times \overline{n-1}}{m^3} \times$

$x^m X^{n-2}$ &c. $= \frac{x^m X^n}{m} - \frac{n}{m} \times \frac{A}{X} - \frac{n-1}{m} \times \frac{B}{X} -$ &c. Q. E. I.

88. If $V = \log. \overline{1-x}$ and it was required to find the fluent of $\dot{x}x^{m-1}V$ *. Suppose $\dot{z} = \dot{x}x^{m-1}$, $v = V$ and let $\dot{w}$ (art. 53.) $= \dot{v} = \frac{\dot{x}}{1-x}$. then

* De Moivre Miscell. Analyt. lib. VI.

$z = \frac{1}{m}v^m$ and $z' = \boxed{\frac{\dot{x}x^m}{m\times\overline{1-x}}}$ also $\dot{v}=1$, and $\ddot{v}=0$:

Now $\frac{-\dot{x}x^m}{m\times\overline{1-x}}$ or $\frac{+\dot{x}x^m}{m\times\overline{x-1}} = \frac{\dot{x}x^{m-1}}{m} + \frac{\dot{x}x^{m-2}}{m}$

$\ldots\ldots \frac{\dot{x}}{m\times\overline{x-1}} = \frac{\dot{x}x^{m-1}}{m} + \frac{\dot{x}x^{m-2}}{m} \ldots -\frac{\dot{V}}{m}$

whose fluent $= z' = \frac{x^m}{m^2} + \frac{x^{m-1}}{m\times\overline{m-1}} \ldots -\frac{V}{m}$.

And therefore the fluent of $V\dot{x}x^{m-1} = zv - z'\dot{v} =$
$\frac{x^mV}{m} + \frac{x^m}{m^2} + \frac{x^{m-1}}{m\times\overline{m-1}} \ldots -\frac{V}{m}$. Q. E. I.

Of Exponential Quantities.

89. When a quantity has a variable index it is called an exponential quantity, as y^x where y and x are variable quantities, or a^x where x only is variable, it is an exponential quantity of the first degree when the index is a simple flowing quantity, and an exponential quantity of the second degree when the index itself is an exponential quantity, as v^{y^x}.

90. Let the respective logarithms of y and v be Y and V, then to find the fluxion of y^x put $y^x = z$ and let log. $z = Z$ then $xY = Z$ and taking the fluxions $\dot{x}Y + x\dot{Y} = \dot{Z}$ or since $\dot{Y} = \frac{\dot{y}}{y}$ and $\dot{Z} = \frac{\dot{z}}{z}$ the modulus being supposed $=1$, we have $\dot{x}Y + \frac{x\dot{y}}{y} = \frac{\dot{z}}{z} = \frac{\dot{z}}{y^x}$ and $\dot{z} = \dot{x}y^xY + x\dot{y}y^{x-1}$.

In the same manner to find the fluxion of vy^x or v^z put it $= w$ then $\dot{w} = \dot{z}v^z\text{V} + z\dot{v}v^{z-1}$. But $\dot{z} = \dot{x}y^x\ \text{Y} + \dot{y}xy^{x-1}$ Therefore $\dot{w} = \dot{x}y_x\, w^x\ \text{VY} + \dot{y}xy^{x-1}\ vy^x\text{V} + \dot{v}y^x\ vy^x{-1}$.

In like manner the fluxion of $a^x = \dot{x}a^x\ \text{A}$ where $\text{A} = \log.\ a$ and the fluxion of $a^{y^x} = \dot{x}y^x\ ay^x\ \text{AY} + \dot{y}xy^{x-1}\ ay^x\text{A}$.

Corol. Hence the fluent of $\dot{x}\ a^x = \frac{a_x}{\text{A}}$.

91. Let the modular ratio be that of e to 1, then log. $e = \text{M}$ and log. $e^x = x\text{M}$ and by Prop. XV. $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 +$ &c. And in the same manner $e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$, &c. Whence $\frac{e^x + e^{-x}}{2} = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 +$ &c; and $\frac{e^x - e^{-x}}{2} = x + \frac{1}{6}x^3 + \frac{1}{120}x^5$, &c.

If we change the sign of x^2 then $\frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 +$ &c. and

$$\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \&c.$$

If q be any index, then by Lemma II. $\overline{1 + \frac{x}{q}}|^q = 1 + x + \frac{q-1}{2q}x^2 + \frac{q-1 \times q-2}{6q^2} \times x^3$ &c. Suppose q to be an infinite number, then $q-1 = q-2 = q$,

$= q$, and $\overline{1+\frac{x}{q}}\Big|^{q} = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$ &c. $= e^x$. In the same manner $\overline{1-\frac{x}{q}}\Big|^{q} = 1-x+\frac{1}{2}x^2-\frac{1}{6}x^3$, &c.$= e^{-x}$.

92. If a be a number whose logarithm is A, then log.$a^x = x$A and $a^x = 1+\frac{xA}{M}+\frac{x^2A^2}{2M^2}+\frac{x^3A^3}{6M^3}+$ &c.

Let log. $x =$ X then $x = 1+\frac{X}{M}+\frac{X^2}{2M^2}+\frac{X^3}{6M^3}+$ &c. and $x^2 = 1+\frac{2X}{M}+\frac{2X^2}{M^2}+$ &c. $x^3 = 1+\frac{3X}{M}$ &c. and $x^4 = 1+$&c. and substituting these values $a^x = 1+\frac{A}{M}+\frac{2AX+AA}{2M^2}\frac{3AX^2+6A^2X+A^3}{6M^3}+$&c.

If $a=x$, then $x^x = 1+\frac{X}{M}+\frac{3X^2}{2M^2}+\frac{10X^3}{6M^3}+$ &c.

Since log. $x^x = x$X and $x = 1+\frac{X}{M}+\frac{X^2}{2M^2}+$ &c. we have log. $x^x = X+\frac{X^2}{M}+\frac{X^3}{2M^2}+$ &c. $= M\times\frac{X}{M}+\frac{X^2}{M^2}+\frac{X^3}{2M^3}$ &c. which compared with the series in art. 86. gives the number $x^x = 1+\frac{X}{M}+\frac{2+1}{2M^2}X^2+\frac{3+6+1}{6M^3}\times X^3$ &c.$= 1+$

$\frac{X}{M}$

$\frac{X}{M} + \frac{3X^2}{2M^2} + \frac{10X^3}{6M^3} +$ &c. which is the same as we found above.

In the same manner log. $x^{x^x} = x^x \times \log.x = X + \frac{X^2}{M} + \frac{3X^3}{2M^2}$ &c. $= M \times \frac{X}{M} + \frac{X^2}{M^2} + \frac{3X^3}{2M^3}$ &c.

whose number by art. 86 is $= 1 + \frac{X}{M} + \frac{3X^2}{2M^2} + \frac{16X^3}{6M^3} +$ &c. $= x^{x^x}$.

The measure of any ratio as of $a + y$ to a the modulus being M is thus expressed $M\left|\frac{a+y}{a}\right.$.

Prop. XVII.

To express fluents by the measures of ratios.

93. By art. 79 the fluent of $\frac{M\dot{y}}{a+y}$, is $M\left|\frac{a+y}{a}\right.$, therefore the fluents of all quantities that can be referred to this form, may be expressed by the measures of ratios.

Example I.

To find the fluent of $\frac{d\dot{x}x^{n-1}}{e+fx^n}$. make $fx^n = N$, then $nf\dot{x}x^{n-1} = \dot{N}$, $d\dot{x}x^{n-1} = \frac{d}{nf}\dot{N}$ and $\frac{d\dot{x}x^{n-1}}{e+fx^n} = \frac{d}{nf} \times \frac{\dot{N}}{e+N}$ which compared with the above form gives $a = e$, $M = \frac{d}{nf}$, $y = N$, and the fluent is $\frac{d}{nf}\left|\frac{e+N}{e}\right. = \frac{d}{nf}\left|\frac{e+fx^n}{e}\right.$.

94. By art. 81. The fluent of $\frac{2aM\dot{x}}{a^2 - x^2}$ is $M \left| \frac{a+x}{a-x} \right.$, and in the same manner the fluent of $\frac{2aM\dot{N}}{N^2 - a^2}$ is $M \left| \frac{N+a}{N-a} \right.$.

Example II.

To find the fluent of $\frac{d\dot{x}x^{\frac{1}{2}n-1}}{e-fx^n}$, make $fx^n = N^2$ then $nf\dot{x}x^{n-1} = 2N\dot{N}$ and $2\dot{N} = \frac{nf\dot{x}x^{n-1}}{N} = \frac{nf\dot{x}x^{n-1}}{x^{\frac{1}{2}n}\sqrt{f}} = n\dot{x}x^{\frac{1}{2}n-1} \times \sqrt{f}$ and therefore $\frac{d\dot{x}x^{\frac{1}{2}n-1}}{e-fx^n} = \frac{2d}{n\sqrt{f}} \times \frac{\dot{N}}{e-N^2}$ which compared with the above gives $a^2 = e$, $a = \sqrt{e}$, $2aM = \frac{2d}{n\sqrt{f}}$, $M = \frac{d}{n\sqrt{ef}}$,

And the fluent is $\frac{d}{n\sqrt{ef}} \left| \frac{\sqrt{e}+\sqrt{fx^n}}{\sqrt{fx^n}-\sqrt{e}} \right.$

or $\frac{d}{ne}\sqrt{\frac{e}{f}} \left| \frac{\sqrt{\frac{e}{f}} + x^{\frac{1}{2}n}}{x^{\frac{1}{2}n} - \sqrt{\frac{e}{f}}} \right.$. Make $R = \sqrt{\frac{e}{f}}$, $T = x^{\frac{1}{2}n}$,

$SS = TT - RR = x^n - \frac{e}{f}$, then the fluent will be $\frac{d}{ne} R \left| \frac{R+T}{T-R} \right.$, or since the ratio of $R+T$ to $T-R$ or $\overline{R+T}|^2$ to $TT-RR$ or SS is double the ratio of $R+T$ to S, the fluent will $\frac{2}{ne} dR \left| \frac{R+T}{S} \right.$.

Example

Example III.

To find the fluent of $\frac{d\dot{x}x^{\frac{1}{2}n-1}}{\sqrt{e+fx^n}}$: make $\frac{e+fx^n}{x^n} = \frac{e}{x^n}+f = N^2$, then $\frac{-ne\dot{x}x^{-1}}{x^n} = 2N\dot{N}$ and $\dot{x}x^{-1} = \frac{-2}{ne}\times N\dot{N}x^n = \left(\text{Since } x^n = \frac{e}{NN-f}\right) = \frac{-2}{n}\times\frac{\dot{N}N}{NN-f}$. and $\frac{d\dot{x}x^{\frac{1}{2}n-1}}{\sqrt{e+fx^n}} = \frac{-2d}{n}\times\frac{\dot{N}}{N^2-f} = \frac{2d}{n}\times\frac{\dot{N}}{f-N^2}$ which may be compared with the fluxion in Example II. where $d=\frac{2d}{n}$, $n=2$, $e=f$, and $f=1$, $R=\sqrt{f}$, $T=N=\sqrt{\frac{e+fx^n}{x^n}}$, $S=\sqrt{\frac{e}{x^n}}$ and the fluent is $\frac{2}{nf}dR\Big|\frac{R+T}{S}$.

Example IV.

To find the fluent of $\frac{d\dot{x}x^{\frac{1}{2}n-1}}{e-fx^n}$ This fluxion by division becomes, $\frac{-d}{f}\dot{x}x^{\frac{1}{2}n-1}+\frac{de}{f}\times\frac{\dot{x}x^{\frac{1}{2}n-1}}{e-fx^n}$. The fluent of the first part is $\frac{-2d}{nf}x^{\frac{1}{2}n}$, and the fluent of the second part by Ex.II. is $\frac{2}{ne}\times\frac{de}{f}\times R\Big|\frac{R+T}{S} = \frac{2}{nf}dR\Big|\frac{R+T}{S}$, where $R=\sqrt{\frac{e}{f}}$, $T=x^{\frac{1}{2}n}$,

S=

$S = \sqrt{\frac{-e+fx^n}{f}}$, therefore the whole fluent is $-\frac{2}{nf} dx^{\frac{1}{2}n} + \frac{2}{nf} dR \left| \frac{R+T}{S} \right.$:

Example V.

To find the fluent of $d\dot{x}x^{-1}\sqrt{e+fx^n}$. make $e+fx^n = NN$, then $2N\dot{N} = nf\dot{x}x^{n-1}$ and $\dot{x}x^{-1} = \frac{2}{nf} \times \frac{N\dot{N}}{x^n} = \left(\text{because } x^n = \frac{N^2-e}{f}\right) \frac{2}{n} \times \frac{N\dot{N}}{N^2-e}$ and therefore $d\dot{x}x^{-1}\sqrt{e+fx^n} = \frac{2d}{n} \times \frac{N^2\dot{N}}{N^2-e} = -\frac{2d}{n} \times \frac{N^2\dot{N}}{e-N^2}$ which may be compared with that in Exam. IV. where $d = \frac{-2d}{n}$, $n=2$, $e=e$, $f=1$, $R = \sqrt{e}$, $T = N = \sqrt{e+fx^n}$, $S = \sqrt{-e+N^2} = \sqrt{fx^n}$, and the fluent is $\frac{2d}{n} N - \frac{2d}{n} R \left| \frac{R+T}{S} \right. = \frac{2d}{n}\sqrt{e+fx^n} - \frac{2d}{n} R \left| \frac{R+T}{S} \right.$.

6. In the same manner the fluent of $d\dot{x}x^{\frac{1}{2}n-1}\sqrt{e+fx^n}$ is $\frac{d}{n} x^n N + \frac{e}{nf} dR \left| \frac{R+T}{S} \right.$ where $R = \sqrt{f}$, $T = \sqrt{\frac{e+fx^n}{x^n}}$, $S = \sqrt{\frac{e}{x^n}}$ and $N = T$.

7. The

7. The fluent of $d\dot{x}x^{\frac{1}{2}n-1}\sqrt{e+fx^n}$ is $\frac{dx^nN}{4nf} \times \overline{2N^2x^n-e} - \frac{e^2}{4nff}dR\left|\frac{R+T}{S}\right. = \frac{e+2fx^n}{4nf}dx^nN - \frac{e^2}{4nff}dR\left|\frac{R+T}{S}\right.$.

8. And the fluent of $d\dot{x}x^{-\frac{1}{2}n-1}\sqrt{e+fx^n}$ is $-\frac{2}{n}dN + \frac{2}{n}dR\left|\frac{R+T}{S}\right.$.

Those who would see more of this kind may consult the excellent Mr. Cotes's *Harmonia Mensurarum*, Part. II.

SECT. IV.

Of curve lines and drawing of tangents.

Definitions.

95. Let AD (fig. 3.) be a right line given in poſition, and the point A in it alſo given; and ſuppoſe a variable line as OP to move always parallel to itſelf, ſo that its extremity P may trace out the curve DP, then the right line OP is called an *Ordinate* and the line AO the diſtance of the ordinate from the given point A is called the *abſciſſa*.

An equation expreſſing the relation between the ordinate and abſciſsa is called the equation of the curve.

If the relation of the ordinate and abſciſsa can be expreſſed by an equation which involves only the ordinate and abciſsa together with invariable known quantities, then the curve is called an algebraic or geometrical curve; but if it cannot be ſo expreſſed it is called a mechanical or tranſcendental curve.

If neither the ordinate or abſciſſa ariſes to above one dimenſion in the equation of the curve line, then the line is called a line of the firſt order.

If they arrive at two dimenſions it is a line of the ſecond order; and ſo on.

96. By altering the beginning of the abſciſſa, the number of dimenſion of the equation will not be altered. For let C (fig. 3.) be the new beginning of the abſciſs, and call the ordinate OP, y, the firſt abſciſſa AO, x, the new abciſſa CO, z, and AC

AC, a. Then will AO=AC+CO, or $a+z=x$, and substituting this value for x in the equation, the resulting equation will be of the same dimensions as the former equation, because $a+z$ and x are of the same number of dimensions and so also are their powers.

97. By altering the angle AOP, which the ordinate contains with the base AO, into the angle ANP, the number of dimensions in the equation will not be altered. For the triangle ONP will remain always similar to itself, and therefore NP is to OP in a given ratio, and NO to OP also in a given ratio, call AO, x, OP.y, AN, z, NP.v and let NP : OP :: 1 : m and NO : OP :: n : 1, then OP$=y=mv$, and NO$=ny=mnv$, and $x=$ AN+NO$=z+mnv$, therefore if we substitute these values of x and y in the given equation, the new equation will be of the same dimensions as the former.

Scholium. By this means we may transform any geometrical curve into another of the same order *.

98. As every geometrical curve may be described by the motion of a point according to a certain constant law, it is evident that this motion must continue for ever, that is, the curve must either return into itself, or go off *ad infinitum*, with a continued curvature, or two arcs of the curve must touch one another.

Hence if the curve has one infinite arc, it must also

* Newton. Princip. Lib. I. Lem. XXII.

have

have another, for otherwise the describing point would stop.

Hence likewise the number of infinite arcs is always even *.

PROP. XVIII.

A line of the first order, or which is the same thing, the locus of a simple equation is always a right line.

99. For the general equation of one dimension is of this form $y - ax + b = 0$. On the right line AB (fig. 4.) take $AC = \frac{b}{a}$ and through C, draw the right line CP making the angle PCO with ACB, whose cosine shall be to its sine as 1 to a, then will CP be the locus of this equation, for take any point P in CP and let fall PO perpendicular to AB and call AO, x, OP, y then by construction CO : OP :: 1 : a, and $CO = \frac{y}{a}$. And $x = AC + CO = \frac{b+y}{a}$, or $ax = b + y$ and $y - ax + b = 0$.

PROP. XIX.

100. *A line of the second order is always a conic section.*

The general equation of two dimensions is of this form.
$$\left.\begin{matrix} y^2 + ax \\ + b \end{matrix}\right\} y \left.\begin{matrix} + cx^2 \\ + dx \\ + e \end{matrix}\right\} = 0.$$

* Stirling Lin. 3ti ordin. Prop. I.

Instead

Instead of y substitute $v - \frac{1}{2}ax - \frac{1}{2}b$ and the equation will become, $v^2 - \frac{1}{4}a^2x^2 + cx^2 - \frac{1}{2}abx + dx - \frac{1}{4}bb + e = 0$ or $v^2 = \frac{a^2 - 4c}{4}x^2 + \frac{ab - 2d}{2}x + \frac{bb - 4e}{4}$.

Let AP (fig. 5.) be the common Apollonian parabola whose axis is AO, and let the latus rectum be p, then $p \times AO = OP^2$. Or if we call AO, x, OP, y, $px = yy$; instead of x put $z + q$ then $yy = pz + pq$.

Let APB (fig. 6.) be an ellipsis, AB the axis, let the latus rectum $= p$, and transverse axis $AB = \frac{p}{r}$ then $\frac{p}{r} : p :: AO \times OB : OP^2$ or $1 : r :: \frac{px}{r} - x^2 : yy$, and $yy = px - rx^2$. Put $x = z + q$ then $yy = -rz^2 + \overline{p - 2qr} \times z + pq - rq^2$.

If the curve was an hyperbola (fig. 7.) then r will be negative and the equation will be $yy = rz^2 + \overline{p + 2qr} \times z + pq + rq^2$.

By comparing the equations of these curves with the above general one of two dimensions, it appears that according as $\frac{aa - 4c}{4}$ is nothing, negative or affirmative the locus will be a parabola, ellipsis or hyperbola; that is, according as $\frac{a^2}{4}$ the square of half the coefficient of xy in the equation, is equal, less, or greater than c the coefficient of x^2.

If x^2 is wanting in the equation then $c=0$, and as $\frac{1}{4}a^2$ is always affirmative, the locus in this case will be an hyperbola.

If xy be wanting then the locus will be a parabola, ellipse, or hyperbola, according as c is nothing, affirmative or negative.

101. Let AQB (fig. 9.) be a circle whose diameter is AB, and center G. Take two right sines OM, oQ at equal distances each side from the center, draw the right line AQ, cutting OM in P. Then the curve which passes through the several points P,p, is called the *Cissoid* of *Diocles*, the point A is called the vertex, the circle AQB the generating circle, and the right BR perpendicular to AB the asymptote.

Call AB, a; AO, x; OP, y then $AO^2 : OP^2 :: Ao^2 : oQ^2$ or $xx : yy :: \overline{a-x}^2 : x \times \overline{a-x} :: a-x : x$ and $yy = \frac{x^3}{a-x}$ or $y = \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}$, which is the equation to this curve.

102. Let the right line GS (fig. 8.) and the point F without it be given in position, and let the right line FP revolve about the point F, and let the lines PR, pR contained between the right line GS and the points P, p, be always equal to a given line AG, then the points P,p will describe a curve which is called the *Conchoid* of *Nicomedes*.

Let the right line FA be perpendicular to GS then the points A, a, are called the vertices of the conchoid, the point F is called the pole.

It is evident the curve can never meet with the line GS which is therefore called the asymptote.

Call

Call the distance AG or PR, a, FG, b GO. x, OP. y, and let fall the perpendicular PQ on GS, then the triangles OPF, QRP are similar and $FO^2 : OP^2 :: PQ^2 : QR^2$ or $\overline{b+x}|^2 : yy :: xx : aa - xx$ and $yy = \dfrac{\overline{b+x}|^2 \times \overline{aa - xx}}{xx}$ and $y = \dfrac{b+x}{x} \times \sqrt{\overline{aa - xx}}|$ which is the equation to the conchoid.

103. Those curves whose ordinates are directly as any power of the abcissa's are called parabola's, and those whose ordinates are reciprocally as any power of the abscissa's are called hyperbola's. Thus if $y = ax^3$, the curve is called a cubical parabola. If $y^2 = ax^3$ or $y = a^{\frac{1}{2}} x^{\frac{3}{2}}$ it is called a Neilian or semicubical parabola. If $y = \dfrac{a}{x^2}$ it is called a cubical hyperbola. And in general if $y = ax^m$, it is a parabola if m is an affirmative number, and an hyperbola if it is negative.

Prop. XX.

104. *A curve line cannot be cut by a right line in more points than there are dimensions in its equation.*

For let the curve PQR (fig. 10.) be cut by the right line OR in the points P, Q, R, and since the equation expresses the relation between the abscissa AO, and each of the ordinates OP, OQ, OR, it is evident that every one of these ordinates will be roots of the equation, and it is proved in Algebra that an equation cannot have more roots, than it has dimensions.

Corol. 1. Hence a curve line whose equation is of an odd number of dimensions must at least cut a right line in one point, since such an equation must have at least one possible root.

Corol. 2. A tangent cannot cut a curve in more points than the number of dimensions diminished by two: thus a tangent to a conic section cannot at all cut the curve. A tangent to a line of the third order may cut it in one point, a tangent to a line of the fourth order may cut it in two points, and so on. Because a tangent may be conceived to cut a curve in two points infinitely near each other.

Corol. 3. Because the number of impossible roots in any equation is always even, a tangent to a curve whose equation is of any odd number of dimensions must cut the curve in at least one point; and a tangent to a curve line, whose equation is of an even number of dimensions, must either not cut the curve at all, or cut it in two points.

Corol. 4. The two last corollaries are also true of the asymptotes of curve lines, which may be considered as tangents at an infinite distance.

105. A tangent to a curve line is a right line drawn through a point in a curve, so near the curve that no right line can be drawn between them.

Let AP (fig. 12.) be a curve whose base is AO, let PT be a tangent in the point P, and OP an ordinate at the same point, then OT the distance between

tween the Points O and T where the ordinate and tangent meet the base is called the *subtangent*; and if at the point P the line PD be drawn perpendicular to the tangent, the line OD is called the *subnormal*.

106. Because no right line can be drawn to the point of contact between the curve and tangent, it is evident that the angle of contact contained between the curve and tangent must be infinitely small, or less than any finite angle.

107. Let the right line PR (fig. 11.) touch the curve PQ in the point P, and let the points Q and R continually approach the point P and at last coincide with it, then the ultimate ratio of the arc PQ to the tangent PR is a ratio of equality.

For draw PS and RQS meeting in a remote point S, then when R and P coincide, the lines SP and SR will coincide, and as the angle RPQ is infinitely small (art. 106.) RQ will be infinitely less than PR or PQ and therefore R and Q will coincide and PR will be equal to PQ.

Prop. XXI.

To draw a Tangent to any Curve.

108. Let AP (fig. 12.) be the curve, OP an ordinate, PT a tangent at the point P, meeting OA in T. Let *op* be an ordinate infinitely near OP, and draw P*n* parallel to AO, call AO, *x*, OP, *y*, then the triangles TOP, P*np* will be similiar and

TO : OP :: Pn : np :: $\dot{x}$: $\dot{y}$ therefore the subtangent TO $= \frac{\dot{x}}{\dot{y}} \times$ OP $= \frac{y\dot{x}}{\dot{y}}$.

Also the triangles DOP, POT are similiar and DO : OP :: OP : OT :: $\dot{y}$: $\dot{x}$ and the subnormal DO $= \frac{y\dot{y}}{\dot{x}}$.

Therefore from the given equation of the curve find the relation of the fluxions $\dot{x}$ and $\dot{y}$, and set off the quantity OT or $\frac{y\dot{x}}{\dot{y}}$ from O towards A if it is affirmative, or the contrary way if it is negative, then draw TP which will touch the curve in the point P.

To draw a perpendicular find $\frac{y\dot{y}}{\dot{x}}$ or OD and set it off from the contrary way from A if it is affirmative, but if it is negative the same way, and draw DP.

Example I. *Fig*. 6.

To draw a tangent to an ellipsis.

109. Let the latus rectum be a, the transverse axis AB $= \frac{a}{b}$. And the equation is (art. 100.) $yy = ax - bxx$, taking the fluxions we have $2y\dot{y} = a\dot{x} - 2bx\dot{x}$, and $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a - 2bx}$, and the subtangent

tangent TO $= \frac{y\dot{x}}{\dot{y}} = \frac{2yy}{a-2bx} = \frac{2ax - 2bxx}{a-2bx}$.

And the subnormal OD $= \frac{y\dot{y}}{\dot{x}} = \frac{1}{2}a - bx$, and the perpendicular PD $= \sqrt{\overline{y^2 + OD^2}} = \sqrt{\frac{1}{4}aa + y^2 \times \overline{1-b}}$. If we make b negative, the curve will be an hyperbola and the subtangent OT (fig. 7.) $= \frac{2yy}{a+2bx}$ $= \frac{2ax+2bx^2}{a+2bx}$, the subnormal OD $= \frac{1}{2}a + bx$ and the perpendicular PD $= \sqrt{\frac{1}{4}aa + y^2 \times \overline{1+b}}$.

And if $b = 0$ the curve will be a parabola, and the subtangent OT (fig. 5.) $\frac{2yy}{a} = 2x$, the subnormal OD $= \frac{1}{2}a$, and PD $= \sqrt{\frac{1}{4}aa + y^2}$.

110 *Corol.* 1. Since OT $= \frac{2ax-2bxx}{a-2bx} = \frac{x \times \overline{\frac{a}{b} - x}}{\frac{a}{2b} - x}$ $= \frac{AO \times OB}{SO}$, we have this proportion SO : AO :: OB : OT and disjointly SO : AO :: BS : AT, and compounding SO : AS :: BS : ST :: AS : ST.

Corol. 2. Since the subnormal OD $= \frac{1}{2}a - bx =$ $b \times \overline{\frac{a}{2b} - x} = b \times$ SO, we have $1 : b$ :: SO : OD or as the transverse axis : latus rectum :: SO : OD.

Corol. 3. Let Q (fig. 46.) be the focus, and let fall the perpendicular QY on the tangent, let SN be the conjugate diameter to SP and let it meet QP in L and PD produced in F, then since the triangles QPY, PLF are similiar, QP : QY :: LP : PF, but by conic sections LP = AS, and PF × PD = SG^2 therefore QP : QY :: AS : $\frac{SG^2}{PD}$ :: PD : $\frac{SG^2}{AS}$ :: PD : $\frac{1}{2}a$ And PD = $\frac{\frac{1}{2}a \times QP}{QY}$.

Example II. *Fig.* 9.

To draw a tangent to the Cissoid *of* Diocles.

111. Let AP be the cissoid, AB the diameter of the generating circle, BE the asymptote, call AB, a; AO, x; OP, y; then by art. 101. $y^2 = \frac{x^3}{a-x}$ and $a - x = \frac{x^3}{y^2}$ and taking the fluxions $-\dot{x} = \frac{3x^2\dot{x}y - 2x^3\dot{y}}{y^3}$ or $-y^3\dot{x} = 3x^2\dot{x}y - 2x^3\dot{y}$ whence $2x^3\dot{y} = 3x^2\dot{x}y + y^3\dot{x}$ and $\frac{\dot{x}}{\dot{y}} = \frac{2x^3}{3x^2y + y^3}$ and the subtangent OT $= \frac{2x^3}{3x^2 + y^2} = \frac{2x^3}{3x^2 + \frac{x^3}{a-x}} = \frac{2x \times \overline{a-x}}{3a - 2x} = \frac{2AO \times OB}{3AB - 2AO} = \frac{2AO \times OB}{AB + 2BO} = \frac{AO \times OB}{AG + BO}$. Therefore as AG + BO : AO :: OB :

OT

OT. The subnormal OD $= \frac{y\dot{y}}{\dot{x}} = \frac{3x^2y^2+y^4}{2x^3} = \frac{y^2}{2x^3} \times \overline{3x^2+y^2} = \frac{3x^2+y^2}{2\times\overline{a-x}} = \frac{3AO^2+OP^2}{2BO}$.

Example III. *Fig.* 8.

To draw a Tangent to the Conchoid *of* Nicomedes.

112. Let AP be a conchoid described with the pole F and distance AG, call AG $=$ PR, a; FG, b; GO, x; OP, y; and QR, v; then $v^2 = a^2 - x^2$, and by art. 102 $y = \frac{\overline{b+x}\times v}{x}$ and taking the fluxions $\dot{y} = \frac{\overline{b+x}\times\dot{v}}{x} - \frac{bv\dot{x}}{x^2}$ But $-\dot{v} = \frac{x\dot{x}}{v}$ therefore $-\dot{y} = \frac{\overline{b+x}\times\dot{x}}{v} + \frac{bv\dot{x}}{x^2}$ and $\frac{\dot{y}}{\dot{x}} = \frac{b+x}{v} + \frac{bv}{x^2} = \frac{xy}{v^2} + \frac{bv}{x^2} = \frac{x^3y+bv^3}{v^2x^2}$ and the subtangent OT $= \frac{y\dot{x}}{\dot{y}} = \frac{v^2x^2y}{x^3y+bv^3} = \frac{xy^2}{\frac{x^2y^2}{v^2}+\frac{bvy}{x}}$. Now since PQ : QR :: FG : GR we have GR $= \frac{bv}{x}$, and since $b+x = \frac{xy}{v}$, FO$^2 = \frac{x^2y^2}{v^2}$, Therefore OT $= \frac{GO\times OP^2}{\overline{FO^2+OP\times RG}}$ And OD which is equal $\frac{PO^2}{OT}$ will be equal $\frac{\overline{FO^2+OP\times RG}}{GO}$.

Let

Let the tangent meet GQ in S, then the triangles SPQ, PTO will be similiar and SQ : QP :: OP : OT and $SQ = \frac{QP \times OP}{OT} = \frac{GO \times OP}{OT} = \frac{FO^2}{OP} + RG.$

Corol. Because $\frac{\dot{y}}{\dot{x}} = \frac{b+x}{v} + \frac{bv}{x^2} = \frac{bx^2 + x^3 + bv^2}{x^2 v}$ and $a^2 = x^2 + v^2$ we have $\frac{\dot{y}}{\dot{x}} = \frac{a^2 b + x^3}{x^2 v}$.

Example IV. *Fig.* 5. 13.

To draw a Tangent to any Parabola or Hyperbola whose equation is $y = ax^m$.

113. By taking the fluxions of this equation we have $\dot{y} = max^{m-1}\dot{x} = \frac{m\dot{x}y}{x}$ and $\frac{\dot{x}}{\dot{y}} = \frac{x}{my}$ therefore the subtangent $\frac{y\dot{x}}{\dot{y}} = \frac{x}{m}$. and the subnormal $\frac{y\dot{y}}{\dot{x}} = \frac{my^2}{x}$.

In the common hyperbola (fig. 13.) $y = \frac{a}{x}$, $m = -1$ and the subtangent $OT = -x = -AP$ and $OD = \frac{-y^2}{x}$

In the cubical parabola $y = ax^3$, $m = 3$, and $OT = \frac{1}{3} AP$.

When the index m is less than 1, and affirmative, the curve will cut the base at right angles in A, because $PT^2 = y^2 + \frac{x^2}{m^2} =$

y^2

$y^2 \times \overline{1+\frac{x^{2-2m}}{m^2a^2}}$ = (when $x=0$) y^2. And the subtangent then is equal to o.

When m is affirmative and greater than 1, the curve will touch the base in A, for then $PT^2 =$ $a^2x^{2m}+\frac{x^2}{m^2} = x^2 \times \overline{\frac{1}{m^2}+a^2x^{2m-2}} =$ (when $x=0$) $\frac{x^2}{m^2}$ and $PT = \frac{x}{m} =$ the subtangent OT.

When m is negative the ordinate AG at A will be an asymptote to the curve, because $PT=y$, and y in that case is infinite.

Of Mechanical Curves.

114. The mechanical curves are those in which the relation of the ordinate and abscissa cannot be expressed by a finite equation, consisting of the powers of the ordinate and abscissa together with known quantities; but is expressed by means of irrational quantities as logarithms, the areas of curves, &c. But to express it by means of only the abscissa and ordinate in finite terms we must do it by means of their fluxions.

Of the Logistica or Logarithmic Curve.

115. If on the right line AB (fig. 14.) there be taken the parts AH, HK, KL, LM, &c. all equal and at the points A, H, K, L, M there be raised the ordinates AV, HW, KX, LY, MZ in geometrical progression, then the curve passing through all the points V, W, X, Y, &c. is called the

the *Logistica* or logarithmic curve. The right line AB is called the asymptote, because as the geometrical progression of the ordinates can never stop, the curve can never meet the line AB.

Draw two ordinates OP, *op* (fig. 15.) infinitely near each other, and let the indefinitely small line O*o* be given, then the ratio between OP and *o*P is given, and by division the ratio between OP and *n*P is given. Let the right line PT be a tangent in the point P then $OT = \frac{OP \times Oo}{Pn}$ and since $\frac{OP}{Pn}$ and O*o* are given, the subtangent OT will also be given.

Call OT, a; AO, x; OP, y; then $a = \frac{y\dot{x}}{\dot{y}}$, and $a\dot{y} = y\dot{x}$ which is the fluxional equation of this curve, and since $a\dot{y} = y\dot{x}$ we have $\dot{x} = \frac{a\dot{y}}{y}$ and taking the fluents by Prop. XIV. Corol. I. $a \left| \frac{AV}{y} \right. = x$, AV being the ordinate at the beginning of the abscissa, or $a \times \log. y = x$, and $\log. y = \frac{x}{a}$, let b be that quantity whose logarithm is $\frac{1}{a}$ then will $\frac{x}{a}$ be $= \log. b^x$ therefore $\log. y = \log. b^x$ and $y = b^x$.

Corol. The subnormal OD is $= \frac{y\dot{y}}{\dot{x}} = \frac{y^2}{a}$. and the perpendicular $DP = \sqrt{yy + \frac{y^4}{aa}} = \frac{y}{a}\sqrt{aa + yy}$.

Lemma

Lemma IV.

116. *Let* AE*e* *(fig.* 16 *) be a circle,* AE *any arc.* CD *its coſine,* DE *its right ſine. I ſay the fluxion of* $AF = \frac{AC}{CD} \times \dot{\overline{DE}}$.

For the triangles C*de*, *e*FE are ſimiliar and C*d* : C*e* :: *e*F : E*e* :: $\dot{\overline{DE}}$: $\dot{\overline{AE}}$ and $\dot{\overline{AE}} = -\frac{Ce}{Cd} \times \dot{\overline{DE}}$

Alſo C*d* : *de* :: *e*F : FE :: $\dot{\overline{DE}}$: $\dot{\overline{DC}}$ and $\dot{\overline{DE}} = \frac{Cd}{de} \times \dot{\overline{CD}}$ Q.E.D.

Of Cycloids.

117. If a circle ARL (fig. 17.) whoſe center is C, goes forward on the right line AB, revolving round its center in the manner of a wheel, then any point as H either within it (fig. 17.) in its its circumference (fig. 18.) or without it (fig. 19.) by its motion in common with the circle will deſcribe a curve as HP, which is called a *Trochoid*, or *Cycloid*, and the circle ARL is called the generating circle, and the right AB the baſe of the cycloid.

Let P be a point in the cycloid, EN the generating circle in that ſituation, F its center, draw FP and let it cut the circle EN in N, draw PO parallel to AB, and about the center C with the radius CH deſcribe the circle KMH. And as all the points of the circumference AR continually apply themſelves to the right line AE, it is plain that the arc EN or AR = AE = OQ = OP + OM. But as the arcs AR, and HM are ſimiliar,

AC

AC : CH :: AR : HM and $HM = \frac{CH \times AR}{AC}$ $= \frac{CH}{AC} \times OQ = \frac{CH}{AC} \times \overline{OP + OM}.$

Call CH, a; CA, b; CO, x; OP, y; OM, v; and the arc HM, z; then we have $z = \frac{a}{b} \times \overline{y+v}$ and taking the fluxions $\dot{z} = \frac{a}{b} \times \overline{\dot{y} + \dot{v}}$ But by Lemma IV. $\dot{z} = \frac{a\dot{x}}{v}$ and $\dot{v} = -\frac{x\dot{x}}{v}$ therefore $\frac{a\dot{x}}{v} = \frac{a}{b} \times \overline{\dot{y} - \frac{x\dot{x}}{v}}$ or $b\dot{x} = v\dot{y} - x\dot{x}$ and $\frac{\dot{x}}{\dot{y}} = \frac{v}{b+x}$. Whence the subtangent $OT = \frac{y\dot{x}}{\dot{y}} = \frac{vy}{b+x} = \frac{Om \times OP}{AO}$, Therefore AO : Om :: PO : OT, but the two right angles AOm, POT are equal, therefore the triangles AOm, POT are similiar and the angle mAO equal to the angle TPO, and if the tangent PT is produced till it meets Am in G, the angle AGP will be a right one, that is, the tangent PT is perpendicular to the line Am, and consequently the perpendicular PD is parallel to Am.

Corol. In the common cycloid $a = b$ and $\dot{x} : \dot{y} :: v : a+x$ whence $\dot{x}^2 : \dot{y}^2 :: v^2 : \overline{a+x}|^2 :: a-x : a+x$ and $\dot{x}^2 + \dot{y}^2 : \dot{y}^2 :: 2a : a+x ::$ AL : AO and $\frac{\dot{y}^2}{AO \times \dot{x}^2 + \dot{y}^2} = \frac{1}{2a} =$ a given quantity.

118. If a circle BPV (fig. 20.) whose center is D, revolves on the circumference of another circle AB, either within or without it, and by a point as P in its circumference it describes a curve AP, this curve is called an *Epicycloid*. In which as the several

ſeveral points of the circular arc BP continually apply themſelves to the arc AB, it is evident than $AB = BP$.

And as the circle turns about the point B in its revolution, it is plain that the right line BP is perpendicular to the curve AP, and that the line VP, which is perpendicular to BP, being produced will be a tangent to the curve AP in the point P.

Of Spirals.

119. If inſtead of expreſſing the curve by the relation of the ordinate and abſciſſa, it is expreſſed by the relation of the right lines SP meeting in a given point S (fig. 21.) and the circular arc BF deſcribed about the center S, with a given radius SB, then the curve is called a *Spiral*.

If the arc BF is as SP, the curve is called the ſpiral of *Archimedes*.

If BF is inverſely as SP, it is called the reciprocal ſpiral.

If BF is inverſely as the ſquare of SP, the curve is called by Mr Cotes the *Lituus*.

120. There are ſome ſpirals which cannot get without a certain circle, and ſome that cannot get within a certain circle, but which continually approach to that circle as an aſymptote. Of the firſt kind is that whoſe equation is $ayx^2 - y^2x^2 = 1$ or $\frac{1}{xx} = ay - yy$ in which y cannot be greater than a. And of the other kind is that whoſe equation is $\frac{1}{xx} = y^2 - ay$ where y cannot be leſs than a.

And

And there are some spirals which are contained within a certain angle, as that whose equation is $y^2 = ax - xx$ where x cannot be greater than a.

PROP. XXII.

To draw a Tangent to a Spiral.

121. Let Spf be a ray infinitely near SPF, with the center S and radius Sp draw the circular arc pn, let PT be a tangent in the point P, draw ST at right angles to SP, and let it meet PT in T. Then the triangles pnP, TSP are similiar (art. 107.) and $Pn : np :: SP : ST$, and as the arcs pn and Ff are similiar, $np : Ff :: Sp$ or $SP : SF$, whence *ex æquo* $Pn : Ff :: SP^2 : ST \times SF$ and $ST = \frac{SP^2 \times Ff}{SF \times Pn}$. Call SF, r, SP, y, and the arc BF, x, then $\frac{Ff}{Pn} = \frac{\dot{x}}{\dot{y}}$ (art. 6.) and the subtangent $ST = \frac{y^2\dot{x}}{r\dot{y}}$.

If SY be let fall perpendicular to the tangent, then $Pp : pn :: SP : SY$ and $Pp : Ff :: SP^2 : SY \times SF$ and $SY = \frac{SP^2 \times Ff}{SF \times Pp}$. call the fluxion of the arc SP, $\dot{t}$ then $\frac{Ff}{Pp} = \frac{\dot{x}}{\dot{t}}$ and $SY = \frac{y^2\dot{x}}{r\dot{t}}$. Where $\frac{\dot{t}}{\dot{x}} = \sqrt{\frac{\dot{y}^2}{\dot{x}^2} + \frac{yy}{rr}}$ or $\dot{t} = \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{r^2}}$ because $Pp = \sqrt{Pn^2 + np^2} = \sqrt{Pn^2 + \frac{SP^2 \times Ff^2}{SF^2}}$.

122.

122. Let the arc BF be as any power m of the ray SP, that is let $x : y^m :: r : a^m$ then by Prop. III. $\dot{x} : m\dot{y}y^{m-1} :: r : a^m$ or $\frac{\dot{x}}{\dot{y}} = \frac{mry^{m-1}}{a^m}$ and the subtangent $\frac{y^2\dot{x}}{r\dot{y}} = \frac{my^{m+1}}{a^m}$.

To find SY we have $\frac{\dot{x}^2}{\dot{y}^2} = \frac{m^2r^2y^{2m-2}}{a^{2m}}$ and $\frac{\dot{t}^2}{\dot{x}^2} = \frac{a^{2m}}{m^2r^2y^{2m-2}} + \frac{y^2}{r^2}$ and $\frac{\dot{x}^2}{r^2\dot{t}^2} = \frac{m^2y^{2m-2}}{a^{2m}+m^2y^{2m}}$ Whence $SY = \frac{y^2\dot{x}}{r\dot{t}} = \frac{my^{m+1}}{\sqrt{a^{2m}+m^2y^{2m}}}$.

In the spiral of Archimedes, $m=1$, $ST = \frac{y^2}{a}$, and $SY = \frac{y^2}{\sqrt{a^2+y^2}}$,

In the reciprocal spiral, $m = -1$, $ST = \frac{-1}{a^{-1}} = a =$ a given quantity and $SY = \frac{1}{\sqrt{a^{-2}+y^{-2}}} = \frac{ay}{\sqrt{a^2+y^2}}$.

In the Lituus, $m = -2$, $ST = \frac{-2y^{-1}}{a^{-2}} = \frac{2a^2}{y}$, and $SY = -\frac{-2y^{-1}}{\sqrt{a^{-4}+4y^{-4}}} = \frac{-2a^2y}{\sqrt{y^4+4a^4}}$.

123. When m is negative, the subtangent will be $\frac{-my^{-m+1}}{a^{-m}} = \frac{-ma^m}{y^{m-1}}$. And when $x=0$, y^{-m} which is $= \frac{a^{-m}x}{r}$ will $= 0$, and therefore y^m and

consequently y will be infinite. Now the subtangent $\frac{-ma^m}{y^{m-1}}$ will be $= 0$, except when $m = 1$. In which case it is equal a, therefore in this case draw SD (fig. 22.) at right angles to SB and equal to a, and draw DE parallel to SB, then will DE touch the curve at an infinite distance, that is, it will be an asymptote to the curve.

But when m is any other number greater than 1, the subtangent being 0, the right SB (fig. 23.) will be an asymptote to the curve.

When m is less than 1, the curve will have no asymptote, because then the subtangent $-ma^m y^{1-m}$ will be infinite.

When m is negative, the curve can never get to the center S until after an infinite number of revolutions, for y cannot $= 0$ unless x is infinite. But when m is affirmative, y will $= 0$ when $x = 0$.

124. If SP (fig. 24.) be such a curve that always makes the same angle with the rays drawn to S, then this curve is called the *equiangular spiral.*

Since in the triangle SPT the angles SPT, TSP are given the triangle SPT will be given in species, and the ratio SP to ST will be given, therefore suppose PS : ST :: $r : a$. and the subtangent ST will $= \frac{ay}{r}$.

In the same manner SPY will be given in species, and SY will be as SP.

Since ST $= \frac{y^2\dot{x}}{r\dot{y}} = \frac{ay}{r}$ we have $y\dot{x} = a\dot{y}$ and

$a\dot{y}$

$\frac{a\dot{y}}{y} = \dot{x}$, therefore by Prop. XIV. Cor. 1. x is the logarithm of y to the modulus a, or $-x = a \left| \frac{r}{y} \right.$ if x vanishes when $r = y$. Let B be the point where $y = r$, then the subtangent SE in that point $= a$, and the arc BF will be the measure of the ratio between SB and SP to the modulus SE.

If b be the number whose logarithm is $\frac{1}{a}$, then $\frac{-x}{a} = \text{L}.\ b^{-x}$ or log. $b^{-x} = \text{L}.\ y$ and $b^{-x} = y$. or $\frac{1}{b^x} = y$. This curve can never get to S until after an infinite number of revolutions because b^{-x} can never vanish unless x is infinite.

Of the Quadratrix.

125. Let AF (fig. 25.) be a circle whose center is G, and if the curve AP is defined by an equation expressing the relation between the ordinate OP, and the arc AF, then the curve AP is called a *Quadratrix.* In which the point T where the tangent meets the base GO may be thus found.

Call GO, x, OP, y, AG, r, AF, z, GE, t, EF, v. Then since EF and OP are parallel GE : EF :: GO : OP or $t : v :: x : y$ and $ty = vx$ whence $\dot{t}y + t\dot{y} = \dot{v}x + v\dot{x}$. Now by Lemma 4, $\dot{v} = \frac{-t\dot{t}}{v}$, and $\dot{z} = \frac{r\dot{v}}{t} = \frac{-r\dot{t}}{v}$ whence $\dot{t} = \frac{-v\dot{z}}{r}$ and $\dot{v} = \frac{t\dot{z}}{r}$, substitute these values of $\dot{t}$ and

 $\dot{v}$, then

$\dot{v}$ then $-\frac{v\dot{z}y}{r}+t\dot{y}=\frac{t\dot{z}x}{r}+v\dot{x}$, but $t=\frac{vx}{y}$, therefore $\frac{-v\dot{z}y}{r}+\frac{vx\dot{y}}{y}=\frac{vx^2\dot{z}}{ry}+v\dot{x}$, divide by $\frac{v}{ry}$ then $-\dot{z}y^2+rx\dot{y}=x^2\dot{z}+ry\dot{x}$ or $x\dot{y}-y\dot{x}=\frac{\dot{z}}{r}\times\overline{x^2+y^2}$. And $x-\frac{y\dot{x}}{\dot{y}}=\frac{\dot{z}}{r\dot{y}}\times\overline{x^2+y^2}$. But the subtangent OT $=\frac{-y\dot{x}}{\dot{y}}$. Therefore GT $=$ GO $+$ OT $=\frac{\dot{z}}{r\dot{y}}\times$ GP^2. Therefore by the help of the equation between z and y we may find $\frac{\dot{z}}{\dot{y}}$ and consequently GT, and PT being drawn will touch the curve.

Suppose $z=\frac{y^m}{a^{m-1}}$ then $\frac{\dot{z}}{\dot{y}}=\frac{my^{m-1}}{a^{m-1}}$ and GT $=$ $\frac{my^{m-1}}{ra^{m-1}}\times$ GP^2 $=\frac{mz}{ry}\times$ GP^2 $=\frac{m\times \text{AF}\times \text{GP}^2}{\text{AG}\times\text{OP}}$

If $m=1$, that is, if $y=z$, GT $=\frac{\text{GP}^2}{\text{AG}}$

If $m=\frac{1}{2}$, or $y:z^2$ then GT $=\frac{\text{AF}\times\text{GP}^2}{2\text{AG}\times\text{OP}}$

PROP. XXIII.

126. *To draw tangents to curves that are defined by an equation between the right lines* SP, FP *drawn to two points* S *and* F (fig. 70.)

Call SF, $2a$, SP, z, PF, v, let fall the perpendicular PO on SF, and let PT be a tangent in the point

point P, and PD a perpendicular to the curve in the same point, and called SO, x, OP, y, SD, t. Then since $FP^2 = SP^2 + SF^2 - 2SF \times SO$ or $vv = zz + 4aa - 4ax$ we have $x = \frac{zz - vv + 4aa}{4a}$ and $\dot{x} = \frac{z\dot{z} - v\dot{v}}{2a}$. Also since $yy + xx = zz$ we have $y\dot{y} + x\dot{x} = z\dot{z}$ whence $t = x + \frac{y\dot{y}}{\dot{x}} = \frac{z\dot{z}}{\dot{x}} = \frac{2az\dot{z}}{z\dot{z} - v\dot{v}}$ *.

And t being found the subnormal $t - x$ will be known and the subtangent $TO = \frac{yy}{t - x} = \frac{zz - xx}{t - x}$.

We may also find the perpendicular SY let fall on the tangent from the point S. For ST : TD :: SY : DP, and ST : TD :: TD — SD : TD :: $\frac{DP^2}{DO} - SD : \frac{DP^2}{DO}$ therefore SY : DP :: $DP^2 - DO \times SD : DP^2$ and $SY = \frac{DP^2 - DO \times SD}{DP}$. Now $DO = t - x$, and $DO^2 = tt - 2tx + xx$, therefore $DP^2 = OP^2 + DO^2 = tt - 2tx + zz$, and since $DO \times SD = tt - tx$ we have $SY = \frac{zz - tx}{\sqrt{tt - 2tx + zz}}$.

* De Moivre Miscell. Analyt. Lib. 8.

Example.

127. Let the equation be $z + v = 2r$, which denotes an ellipse whose foci are S, F, and whose greater axis is $2r$. Then by this equation we shall have $\dot{z} = -\dot{v}$ and $z\dot{z} - v\dot{v} = 2r\dot{z}$, and $t = \frac{2az\dot{z}}{z\dot{z} - v\dot{v}} = \frac{az}{r}$.

To find SY, we have $x = \frac{zz - vv + 4aa}{4a} = \frac{4rz - 4rr + 4aa}{4a} =$ (If we put $rr - aa = mm$) $\frac{rz - mm}{a}$ whence $tx = \frac{rz^2 - m^2z}{r}$ and $zz - tx = \frac{m^2z}{r}$. Also $t - x = \frac{a^2z - r^2z + m^2r}{ar} = \frac{m^2r - m^2z}{ar}$, and $tt - tx = \frac{m^2rz - m^2zz}{rr}$ therefore $tt - 2tx + zz = \frac{2m^2rz - m^2zz}{rr} = \frac{m^2zv}{rr}$, and $\sqrt{tt - 2tx + zz} = \frac{m}{r}\sqrt{zv}$, whence $SY = \frac{mz}{\sqrt{zv}} = m\sqrt{\frac{z}{v}}$.

In the same manner, the perpendicular Fy let fall from $F = \frac{mv}{\sqrt{zv}} = m\sqrt{\frac{v}{z}}$ and $SY \times Fy = mm$.

PROP.

Prop. XXIV.

128. *To draw a curve of a given kind to touch another given curve in a given point.*

Let AP (fig. 26.) be the given curve, BP another curve touching it in the point P. And let the right line TP touch them both in the same point. By the nature of the curve AP, the subtangent OT will be given, and by assuming a fictitious equation to represent the curve BP we shall have another expression for OT which being made equal to the other value of OT will give the curve BP.

Example I.

Let the curve AP be a circle whose radius $= r$ and let it be required to draw the the parabola BP to touch it in P. Call AO, x OP, y, AB, b. Then $2rx - xx = yy$ and $r\dot{x} - x\dot{x} = y\dot{y}$, or $\frac{\dot{x}}{\dot{y}} = \frac{y}{r-x}$ whence $\text{OT} = \frac{y\dot{x}}{\dot{y}} = \frac{yy}{r-x}$. Assume the equation $c \times \text{AO} = \text{OP}^2$ or $cb + cx = yy$ to represent a parabola, then $c\dot{x} = 2y\dot{y}$ and $\frac{\dot{x}}{\dot{y}} = \frac{2y}{c}$ whence $\text{OT} = \frac{2yy}{c}$ make these two values of OT equal, then $\frac{yy}{r-x} = \frac{2yy}{c}$ or $c = 2r - 2x = 2\text{CO}$, which determines the parabola BP.

Example II.

Instead of the common parabola, let it be required to draw a parabola, or hyperbola whose in

dex is m, to touch the circle AP. Assume $cb + cx = y^m$, then $c\dot{x} = my^{m-1}\dot{y}$ or $\frac{\dot{x}}{\dot{y}} = \frac{my^{n-1}}{c}$ and $TO = \frac{my^m}{c}$ which is to be made $= \frac{y^2}{r-x}$, therefore $c = my^{m-2} \times \overline{r-x} = my^{n-2} \times CO$.

129. If AB becomes a right line, that is, if it is required to draw a curve to touch a right line given in position, Then x will be to y in a given ratio as r to s, and $OT = x = \frac{ry}{s}$, and assuming an equation to the curve and finding OT, if we make the two values of OT equal, we shall find the equation of the curve as before.

As for example, to draw a parabola whose index is m to touch a right line. Assume the equation $bc + cx = y^m$ which will give $OT = \frac{my^m}{c}$ therefore $\frac{my^m}{c} = \frac{ry}{s}$ and $c = \frac{msy^{m-1}}{r}$.

And if c is given and y is to be found we shall have $y^{m-1} = \frac{rc}{ms}$ and $y = \frac{rc}{ms}\Big|^{\frac{1}{m-1}}$.

SECT.

SECT. V.

Of the greatest and least ordinates.

PROP. XXV.

130. *If y be any quantity composed of the powers of a quantity x (that flows uniformly) any how combined with invariable quantities, then supposing the increment of x to be X, I say that in the same time as x becomes $x+X$, y will become*

$$y+\frac{\dot{y}X}{\dot{x}}+\frac{\ddot{y}X^2}{2\dot{x}^2}+\frac{\dddot{y}X^3}{2.3\dot{x}^3}+\frac{\ddddot{y}X^4}{2.3.4\dot{x}^4}+\&c\,*.$$

For suppose $y=Ax^m+Bx^n+Cx^p+$ &c. and let Y be the contemporary increment of y, that is when x becomes $x+X$ let y become $y+Y$, and these values being substituted in the above equation it will become $y+Y=A\times\overline{x+X}|^m+B\times\overline{x+X}|^n+C\times\overline{x+X}|^p+$ &c. and substituting the values of these powers by Lemma II. $y+Y=Ax^m+Bx^n+Cx^p+$ &c. $+$

$$Ax^m\times\frac{mX}{x}+m.\frac{m-1}{2}.\frac{X^2}{x^2}+m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{X^3}{x^3}$$

$$Bx^n\times\frac{nX}{x}+n.\frac{n-1}{2}.\frac{X^2}{x^2}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{X^3}{x^3}$$

$$Cx^p\times\frac{pX}{x}+p.\frac{p-1}{2}.\frac{X^2}{x^2}+p.\frac{p-1}{2}.\frac{p-2}{3}.\frac{X^3}{x^3}$$

But $y=Ax^m+Bx^n+Cx^p$ &c. and by art. 34 $\dot{y}=mA\dot{x}x^{m-1}+nB\dot{x}x^{n-1}+pC\dot{x}x^{p-1}$ &c. $\ddot{y}=m.m-1.$

* Taylor Method. Increm, Prop. VII.

A

$A\dot{x}^2x^{m-2} + n\,.\,\overline{n-1}\,.\,B\dot{x}^2x^{n-2} + p\,.\,\overline{p-1}\,.\,C\dot{x}^2x^{p-2}$ &c. $\dddot{y} = m.\overline{m-1}.\overline{m-2}.\,A\dot{x}^3x^{m-3} + n\,.\,\overline{n-1}\,.\overline{n-2}.$ $B\dot{x}^3x^{n-3} +$ &c. and $\frac{\dot{y}X}{\dot{x}} = Ax^m \times \frac{mX}{x} +$ $Bx^n \times \frac{nX}{x} + Cx^p \times \frac{pX}{x}$ &c. $\frac{\ddot{y}X^2}{2\dot{x}^2} = Ax^m \times$ $m\,.\,\frac{m-1}{2}\,.\,\frac{X^2}{x^2} + Bx^n \times n\,.\,\frac{n-1}{2}\,.\frac{X^2}{x^2} + Cx^p \times$ $p.\frac{p-1}{2}\,.\,\frac{X^2}{x^2} +$ &c. therefore substituting these values $y + Y = y + \frac{\dot{y}X}{\dot{x}} + \frac{\ddot{y}X^2}{2\dot{x}^2} + \frac{\dddot{y}X^3}{2\,.\,3\dot{x}^3} +$ &c. Q. E. D.

131. *Corol.* 1. Let Υ be the decrement of y, and as x flows uniformly its value before it is x, or when $y = y - \Upsilon$ is $x - X$ therefore changing the sign of X we have $y - \Upsilon = y - \frac{\dot{y}X}{\dot{x}} + \frac{\ddot{y}X^2}{2\dot{x}^2} - \frac{\dddot{y}X^3}{2.3\dot{x}^3}$ $+ \frac{\ddddot{y}X^4}{2\,.\,3\,.\,4\dot{x}^4} -$ &c.

Corol. 2. If $\dot{x} = X$, that is, if the increment of x be equal to its fluxion then $y + Y = y + \dot{y} +$ $\frac{1}{2}\ddot{y} + \frac{1}{6}\dddot{y} + \frac{1}{24}\ddddot{y}$ &c. and $y - \Upsilon = y - \dot{y} + \frac{1}{2}\ddot{y} + \frac{1}{6}\dddot{y} +$ $\frac{1}{24}\ddddot{y} -$ &c.

132. An ordinate is said to be a *maximum* or *minimum* when it is a greater or less than each of the ordinates drawn from both the parts of the curve adjoining to that ordinate. It is called a max-

maximum or minimum of the first kind when both these parts are on different sides of the ordinate (fig. 27, 28. 31.)

And it is called a maximum or minimum of the second kind when both the parts are on the same side of the ordinate (fig. 29. 32, 33, 34.)

PROP. XXVI.

To find when an ordinate becomes a maximum or minimum of the first kind.

133. *Case* 1. Let the whole curve be either concave or convex to the abscissa AO (fig. 30.) and if the tangent PT be parallel to AO, the ordinate OP will be either a maximum or minimum, that is, a maximum when the curve is concave, and a minimum when it is convex to the base.

Call AO, x, OP, y, and by art 108, $\dot{y}$ is to $\dot{x}$ as OP is to the subtangent which in the present case is infinite, therefore $\dot{x} : \dot{y} :: \mathrm{OP} : 0$ and $\dot{y}=0$.

To determine whether it is a maximum or minimum or which is the same thing whether the curve is concave or convex towards the base, suppose x to flow uniformly and let $\omega\mathrm{O} = o\mathrm{O} = \dot{x}$, then by prop. XXV. $\omega\pi$ or $y-\Upsilon = y-\dot{y}+\frac{1}{2}\ddot{y}-\frac{1}{6}\dddot{y}+\frac{1}{24}\ddddot{y}$ &c. and op or $y+\mathrm{Y}=y+\dot{y}+\frac{1}{2}\ddot{y}+\frac{1}{6}\dddot{y}+\frac{1}{24}\ddddot{y}$ &c.

Suppose $\dot{y}=0$, Then $y-\Upsilon=y+\frac{1}{2}\ddot{y}-\frac{1}{6}\dddot{y}$ &c. and $y+\mathrm{Y}=y+\frac{1}{2}\ddot{y}+\frac{1}{6}\dddot{y}$. And if $\ddot{y}$ does not vanish, the two ordinates $\omega\pi$, op will be both greater or both less than y or OP according as $\ddot{y}$ is affirmative or negative, or OP is a maximum when $\ddot{y}$ is negative, and a minimum when it is an affirmative.

If

If $\dot{y}$ and $\ddot{y}$ vanish, and at the same time $\dddot{y}$ does not vanish, then $y - Y = y - \frac{1}{6}\dddot{y}$, and $y + Y = y + \frac{1}{6}\dddot{y}$ therefore now one of the ordinates $\omega\pi$, op will be greater, and the other will be less than OP, which in this case cannot be either maximum or minimum.

If $\dot{y}$, $\ddot{y}$ and $\dddot{y}$ vanish and $\ddddot{y}$ does not vanish then the ordinate OP will be a maximum when $\ddddot{y}$ is negative, and a minimum when $\ddddot{y}$ is affirmative.

Therefore if the first fluxion of y together with some of its fluxions of subsequent successive orders vanish, y will be a maximum or minimum when the number of fluxions that vanish is odd, and it it is a maximum when the fluxion of the next order that does not vanish is negative, and a minimum when it is affirmative.

134. *Case* 2. If the tangent PT (fig. 31.) coincides with the ordinate OP, then the ordinate OP will be either a maximum or minimum, if the whole curve be convex or concave to the base. and in this case $\dot{x} : \dot{y} :: 0 :: \text{OP}$ and $\dot{y}$ will be infinite in respect of $\dot{x}$.

To determine whether y is a maximum or a minimum, it is plain that y increases so long as $\dot{y}$ is affirmative, and decreases when $\dot{y}$ is negative, therefore if $\dot{y}$ is affirmative before y comes to OP and is afterwards negative then the ordinate OP is a maximum. If $\dot{y}$ is first negative and then affirmative, y is a minimum. But if $\dot{y}$ has the same sign both before and after y comes to OP, then OP is neither a maximum nor a minimum.

PROP.

PROP. XXVII.

To find when an Ordinate becomes a maximum or minimum of the second kind.

135. *Case* 1. Let the two parts of the curve adjoining to the ordinate OP (fig. 32, 33.) have their concavity turned the same way. Then the tangent PT will cut the base at the greatest or least distance from A, or the line AT will be a maximum or minimum.

Call AO, x; OP, y, then by art. 108. OT $= \frac{y\dot{x}}{\dot{y}}$ and AT $= x - \frac{y\dot{x}}{\dot{y}}$ or $\frac{y\dot{x}}{\dot{y}} - x$. But when AT is a maximum or minimum, its fluxion by the last proposition is either o or infinite, therefore $\dot{x} - \dot{x} + \frac{y\ddot{y}\dot{x}}{\dot{y}^2} = \frac{y\ddot{y}\dot{x}}{\dot{y}^2}$ is either o or infinite, and $\ddot{y}$ is o or infinite.

If the curve passes beyond the ordinate OP, then OP will be neither a maximum nor a minimum.

If the two values of $\dot{y}$ have the same sign before y comes to OP, y will be a maximum if $\dot{y}$ is affirmative and a minimum if it is negative.

If the two values of $\dot{y}$ have different signs, y will be neither a maximum nor a minimum.

136. *Case* 2. If one part of the curve be concave and the other part convex to the base, then the line AT (fig. 91.) will be neither greatest nor least, but of an intermediate magnitude, and the ratio of TO : OP or $\dot{x} : \dot{y}$ will be given.

Call PT, t, and draw TB perpendicular to TP and let fall the perpendicular AB on TB let p be

be any point in the curve, draw the ordinate po to AO and pm perpendicular to TB. Then it is evident that pm is a maximum or minimum, or as tm vanishes when t comes to T, pt must be a maximum or minimum, and its fluxion (by art. 134.) is infinite, or since pt coincides with PT when p comes to P, the fluxion of PT must be infinite, but $\frac{y^2\dot{x}^2}{\dot{y}^2}+y^2=t^2$ or $\frac{\dot{x}^2}{\dot{y}^2}+1=\frac{t^2}{y^2}$ whence taking the fluxions $\frac{-\dot{x}^2\ddot{y}}{\dot{y}^3}=\frac{t\dot{t}}{y^2}-\frac{t^2\dot{y}}{y^3}=$ infinity. Therefore $\ddot{y}$ is infinite, because $\dot{x}$, $\dot{y}$ and y are finite quantities.

The manner of determining whether y in this case is a maximum or a minimum is the same as in the last.

137. To find when any quantity is a maximum or a minimum, we may consider that quantity as the ordinate of a curve, and by the two last propositions find when it is a maximum or minimum.

Example I.

138. *If the product of two quantities be a, to find when their sum is a minimum.*

Let the two quantites be x and $\frac{a}{x}$, then $x+\frac{a}{x}$ is a minimum, make $x+\frac{a}{x}=y$ then $\dot{x}-\frac{a\dot{x}}{x^2}=\dot{y}$ and $+\frac{2a\dot{x}^2}{x^3}=\ddot{y}$, therefore in this case $\ddot{y}$ is affirmative and does not vanish, therefore by prop. XXVI.

XXVI. y is a minimum when $\dot{y}=0$, or $\dot{x}-\frac{a\dot{x}}{x^2}=0$, whence $x^2-a=0$ and $x=\surd a$, and $\frac{a}{x}=\surd a$, therefore the two quantities are each $=\surd a$ and their sum $=2\surd a$.

Example II. *Fig.* 35.

139. *Having given the sum of the legs* AB + BC *of a right angled triangle, to find when the hypothenuse is a minimum.*

Let AB + BC $=a$, AB $=x$ then BC $=a-x$, and $AC^2=x^2+a^2-2ax+x^2=a^2+2x^2-2ax$ which is to be a minimum, therefore x^2-ax must be a minimum. Let $xx-ax=y$, then $2x\dot{x}-a\dot{x}=\dot{y}$, and $2\dot{x}\dot{x}=\ddot{y}$, therefore y is a minimum, when $\dot{y}=0$ or $2x\dot{x}-a\dot{x}=0$; whence $x=\frac{1}{2}a$ $=$ AB, and BC $=\frac{1}{2}a=$ AB and the hypothenuse AC $=\surd\frac{1}{2}a^2=a\surd\frac{1}{2}$.

Example III. *Fig.* 35.

140. *Supposing the hypothenuse given, to find when the area is a maximum.*

Since the area is to be a maximum, AB $\times$ BC will be a maximum, and consequently $AB^2\times BC^2$ will be a maximum Call AC, a, AB, x, then $AC^2=aa-xx$ and $AB^2\times BC^2=a^2x^2-x^4$ which suppose $=2y$ then $a^2x\dot{x}-2x^3\dot{x}=\dot{y}$ and $a^2\dot{x}^2-6x^2\dot{x}^2=\ddot{y}$, put $\dot{y}=0$, then $a^2x\dot{x}-2x^3\dot{x}=0$, and $x^2=\frac{1}{2}a^2$, and $\ddot{y}=\dot{x}^2\times a^2-6x^2=-2a^2\dot{x}^2$, therefore $\ddot{y}$ is negative and consequently y is a maximum, when $x^2=\frac{1}{2}a^2$ or $x=$

$AB = a\sqrt{\frac{1}{2}}$, $BC = a\sqrt{\frac{1}{2}}$ and the area $\frac{1}{2}AB \times BC = \frac{1}{4}aa$.

Example IV. *Fig.* 36.

141. *To divide a right line* AB *into two such parts that the power* m *of the part* AC *multiplyed into the power* n *of the part* CB *or* $AC^m \times CB^n$ *shall be a maximum.*

Call AB, a, AC, x, $CB = z$, then $\dot{z} = -\dot{x}$, let $y = x^m z^n$, then $m\dot{x}z^n x^{m-1} + n\dot{z}x^m z^{n-1} = \dot{y} = 0$, or $m\dot{x}z + nx\dot{z} = m\dot{x}z - nx\dot{x} = \frac{\dot{y}}{x^{m-1}z^{n-1}} = 0$, therefore $mz = nx$ and $x : z :: m : n$.

Taking the fluxions of the above equation $m\dot{x}z - nx\dot{x} = \frac{\dot{y}}{x^{m-1}z^{n-1}}$, putting $\dot{y} = 0$ and $\dot{z} = -\dot{x}$ we have $-m\dot{x}^2 - n\dot{x}^2 = \frac{\ddot{y}}{x^{m-1}z^{n-1}}$. Therefore $\ddot{y}$ is negative and y is a maximum when $AC : CB :: m : n$.

Example V. *Fig.* 38.

142. *To inscribe the greatest rectangle in a given parabola* ABC.

Call AG, a, BC, b, $EK = GH$, x, then $AH = a - x$, and since $AG : AH :: BC^2 : FK^2$ we have $FK^2 = \frac{b^2 \times \overline{a - x}}{a}$ and $FK^2 \times HG^2 = \frac{b^2 x^2 \times \overline{a - x}}{a}$

which

which is to be a maximum, therefore $ax^2 - x^3$ is a maximum, and $2ax\dot{x} - 3x^2\dot{x} = 0$, whence $x = \frac{2}{3}a$, and $AH = \frac{1}{3}AG$. $FK^2 = \frac{1}{3}BC^2$ and the greatest rectangle $FK \times HG = \frac{2}{3} \times AG \times BC \times \sqrt{\frac{1}{3}}$.

Example VI. *Fig.* 37.

143. *Supposing that gravity acts in the direction of lines parallel to* BC, *and that the point* A, *without the right line* BC, *is given in position to find the figure of the curve* APC, *through which a heavy body falling from* A *will arrive at the point* C *in the right line* BC *in the shortest time.*

Let the ordinates OP and *op*, ΩΠ and ωπ be infinitely near each other. Suppose O*o*, Ωω the increments of the abscissa's, and P*n* + Πν the sum of the increments of the ordinates to be given.

Call BO, x, OP, y, ΠΩ, w, $Oo = \Omega\omega$, $\dot{x}$, P*n*, $\dot{y}$, Πν, $\dot{w}$, and the increment of the arc = P*p*, $\dot{t}$. The velocity in O call V and the velocity in Ω, v. Then the time in which P*p* is described is as $\frac{Pp}{V}$ and the time in which Ππ is described is as $\frac{\Pi\pi}{v}$ and their sum $\frac{Pp}{V} + \frac{\Pi\pi}{v} = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{V} + \frac{\sqrt{\dot{x}^2 + \dot{w}^2}}{v}$ is to be a minimum, whence

$$\frac{\dot{y}\ddot{y}}{V\sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{\dot{w}\ddot{w}}{v\sqrt{\dot{x}^2 + \dot{w}^2}} = 0,$$ but since $\dot{y} + \dot{w}$ is given we have $\ddot{y} = -\ddot{w}$, and therefore

$\frac{\dot{y}}{V\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{\dot{w}}{v\sqrt{\dot{x}^2+\dot{w}^2}}$ that is, $\frac{\dot{y}}{V\dot{t}}$ is a given quantity.

Let the velocity V be as $x^{\frac{1}{2}n}$ then $\frac{\dot{y}}{\dot{t}x^{\frac{1}{2}n}}$ or its square $\frac{\dot{y}^2}{\dot{t}^2x^n}$ is a given quantity.

In the common hypothesis of gravity $n=1$ therefore $\frac{\dot{y}^2}{\dot{t}^2x}$ is a given quantity, which is the property of a cycloid whose base is BA and diameter of its generating circle BC by Corol. art. 117.

If $n=2$ then $\frac{\dot{y}}{\dot{t}x}=$ a given quantity as $\frac{1}{r}$ and $\dot{t}=\frac{r\dot{y}}{x}$ which is the property of a circle whose center is B and radius BC by Lemma IV.

If $n=0$, that is, if the velocity be given then $\frac{\dot{y}}{\dot{t}}$ is a given quantity and y is as t, that is, the figure the body describes is a right line.

Example VII. *Fig.* 37.

144. *If the arc* AC, *and the point* A, *without the right line* BC *be given, to find when the area* ABC *is a maximum.*

Call OP, y, ΩΠ, w, Oo, $\dot{x}$, Ωω, $\dot{z}$, Pn, $\dot{y}$, Πν, $\dot{w}$, Pp, $\dot{t}$, Ππ, $\dot{\tau}$ and suppose $\dot{t}+\dot{\tau}$ given, also let Pn=Πν be given, then $y\dot{x}+w\dot{z}$ * is to be a maximum and $y\ddot{x}+w\ddot{z}=0$, but since $\dot{x}^2=\dot{t}^2$

* See Sect. VIII.

$-\dot{y}^2$,

$-\dot{y}^2$, and $\dot{z}^2 = \dot{\tau}^2 - \dot{y}^2$ we have $\ddot{x} = \frac{\dot{t}\ddot{t}}{\dot{x}}$, and $\ddot{z} = \frac{\dot{\tau}\ddot{\tau}}{\dot{z}} = \frac{-\dot{\tau}\ddot{t}}{\dot{z}}$ therefore $\frac{y\dot{t}\ddot{t}}{\dot{x}} = \frac{w\dot{\tau}\ddot{t}}{\dot{z}}$ or $\frac{y\dot{t}}{\dot{x}} = \frac{w\dot{\tau}}{\dot{z}}$, that is, $\frac{y\dot{t}}{\dot{x}}$ is a given quantity, which is the property of a circle whose center is B.

Example VIII.

145. *Supposing that* $\dot{x} = \sqrt{c^2 + \dot{y}^2}$ *to find when* $A\dot{x} - B\dot{y}$ *is a maximum or minimum.*

By taking the fluxion of this quantity we have $A\ddot{x} - B\ddot{y} = 0$ and since $\dot{x}^2 = c^2 + \dot{y}^2$ we have $\ddot{x} = \frac{\dot{y}\ddot{y}}{\dot{x}}$ therefore $\frac{A\dot{y}\ddot{y}}{\dot{x}} - B\ddot{y} = 0$. or $A\dot{y} = B\dot{x}$.

Corol. 1. Hence $\boxed{A\dot{x}} - \boxed{B\dot{y}}$ will be a maximum or minimum when $A\dot{y} = B\dot{x}$, because $\boxed{A\dot{x}} - \boxed{B\dot{y}}$ may be be considered as the sum of all the $A\dot{x} - B\dot{y}$.

Corol. 2. If either of the fluents of $A\dot{x}$ or $B\dot{y}$ be given, the other will be a maximum or minimum when $A\dot{y} = B\dot{x}$.

Suppose the arc t given to find when the solidity or $\boxed{y^2\dot{x}}$ * is a maximum. A will $= y^2$ and the equation of the curve will be $B\dot{x} = y^2\dot{t}$.

If the distance of the point A from the right line BC (fig. 37.) be given, or if the fluent of $\dot{y}$ be given the time of descent from A to C or $\boxed{\frac{\dot{t}}{V}}$

* See Sect. 10.

 will

will be a minimum when $B\dot{t} = \frac{\dot{y}}{V}$ or $B = \frac{y}{V\dot{t}}$, as in art. 143.

If the surface or $\boxed{2y\dot{t}}$* be given the solidity or $\boxed{y^2\dot{x}}$ will be a maximum when $y^2\dot{t} = 2Ay\dot{x}$ or $\dot{t} = \frac{2A\dot{x}}{y}$ which shews that the figure must be a globe.

PROP. XXVIII.

146. *Let* AP (*fig.* 39.) *be a given curve, and* E *a point that is not in the curve, given in position; to find the point* P *in the curve, so that the right line* EP *shall be a maximum or a minimum, or to draw a perpendicular to a curve from a given point.*

Let A be the beginning of the abcissa, let fall the perpendicular EG on AO; call AG, a, GE, b, AO, x, OP, y, draw FP parallel to AO, then $EF = b - y$, $GO = FP = x - a$, and $EP^2 = EF^2 + FP^2$ is to be a maximum or minimum, that is $bb - 2by + yy + xx - 2ax + aa$ is to be a maximum or minimum, therefore $y\dot{y} - b\dot{y} + x\dot{x} - a\dot{x} = 0$ and $\dot{x} \times \overline{x - a} = \dot{y} \times \overline{b - y}$. By this equation we may exterminate either x or y out of the equation of the curve, and so determine the point P. Q. E. I.

Example.

Let AP be a parabola, the point G the focus, and let GE : AG :: 10 : 1, then $b = GE = 10a$, and the above equation will be $\dot{x} \times \overline{x - a} = \dot{y} \times \overline{10a - y}$

* Sect. 11.

or

or $\frac{\dot{x}}{\dot{y}} = \frac{10a - y}{x - a}$. But by the equation of the parabola $4ax = yy$, we have $\frac{\dot{x}}{\dot{y}} = \frac{y}{2a}$ therefore

$$\frac{y}{2a} = \frac{10a - y}{x - a} = \frac{10a - y}{\frac{y^2}{4a} - a} = \frac{40a^2 - 4ay}{y^2 - 4a^2}$$ and

$y^3 - 4a^2y = 80a^3 - 8a^2y$ or $y^3 + 4a^2y - 80a^3 = 0$, which divided by $y^2 + 4ay + 20a^2$ gives $y - 4a = 0$ and $y = 4a$ therefore $x = \frac{y^2}{4a} = 4a$. Which determines the point P.

147. *Corol.* Since $\dot{x} \times \overline{x - a} = \dot{y} \times \overline{b - y}$ we have $\dot{x} : \dot{y} :: b - y : x - a ::$ EF : FP :: PO : OD and $\text{OD} = \frac{y\dot{y}}{\dot{x}}$ as we found in art. 108.

Sect. VI.

Of the Curvature of Curve Lines.

148. THE curvature of any curve, is how much that curve is bent from a right line, and therefore may be meaſured by the angle of contact between the curve and tangent.

In a circle the angle of contact is the ſame throughout the whole circle, and conſequently the curvature is the ſame in all points of the ſame circle. The curvature of different circles may be compared together by the following propoſition.

Prop. XXIX.

149. *The curvatures of different circles are inverſely as the radii of the circles.*

Let AB, FG (fig. 40.) be two circles, C, E, their centers, AD, FH tangents in the points A,F, take the indefinitely ſmall arc AB equal to the arc FG, and let fall the perpendiculars BD, GH on the tangents AD, FH. The angle of contact DAB is to the angle of contact HFG as $\frac{DB}{AB}$ to $\frac{HG}{FG}$ :: DB : HG :: $\frac{AB^2}{2AC}$: $\frac{FG^2}{2EF}$ :: EF : AC and conſequently the curvature of AB is to the curvature of FG as EF : AC that is inverſely as the radii. Q. E. D.

150. If a circle be drawn touching a curve ſo cloſe that no other circle can be drawn between them,

them, that circle will have the same curvature as the curve, and is called the circle of osculation, or circle of curvature, its radius is called the radius of curvature and center the center of curvature.

Therefore to determine the curvature of any curve, it is only required to find a circle equally curve, or to find the radius or center of curvature.

Lemma V.

151. *Let* APp *(fig. 43.) be any curve,* OP, *op, two ordinates infinitely near each other,* PT, *pt, two tangents in the points* P, *p, to find the ultimate ratio of* T*t* : P*p*.

Call the abscissa AO, x, the ordinate OP, y, the subtangent OT, v, the arc AP, t, and put $\frac{\dot{y}}{\dot{x}} = z$, then $ot - OT = tT + Oo$, but (by art. 6.) $ot - OT : Oo :: \dot{\overline{OT}} : \dot{\overline{AO}}$, therefore $tT + Oo : Oo :: \dot{v} : \dot{x}$ and disjointly $tT : Oo :: \dot{v} - \dot{x} : \dot{x}$, But $Oo : Pp :: \dot{x} : \dot{t}$ whence $tT : Pp :: \dot{v} - \dot{x} : \dot{t}$. Now $v = \frac{y}{z}$ and $\dot{v} = \frac{\dot{y}}{z} - \frac{y\dot{z}}{z^2} = \dot{x} - \frac{y\dot{z}}{z^2}$ therefore $\dot{v} - \dot{x} = \frac{-y\dot{z}}{z^2}$ and $tT : Pp :: \frac{-y\dot{z}}{z^2} : \dot{t}$

Q. E. I.

Corol. 1. Let fall the perpendicular TK on pt then $TK : tT :: OP : PT :: \dot{y} : \dot{t}$ and $TK : Pp :: \frac{-y\dot{y}\dot{z}}{z^2} : \dot{t}^2 :: -y\dot{x}^2\dot{z} : \dot{y}\dot{t}^2$.

Corol. 2. Since $Pp^2 = Pn^2 + np^2$ we have $\dot{t}^2 = \dot{x}^2 + \dot{y}^2$ or $\frac{\dot{t}^2}{\dot{x}^2} = 1 + \frac{\dot{y}^2}{\dot{x}^2} = 1 + z^2$ whence taking the fluxions $\frac{\dot{t}\ddot{t}\dot{x} - \dot{t}^2\ddot{x}}{\dot{x}^3} = z\dot{z} = \frac{\dot{y}\dot{z}}{\dot{x}}$ and $\frac{\dot{t}\ddot{t}\dot{x} - \dot{t}^2\ddot{x}}{\dot{y}} \times y = y\dot{x}^2\dot{z}$ whence TK : Pp :: $\frac{\dot{t}\ddot{t}\dot{x} - \dot{t}^2\ddot{x}}{\dot{y}} \times y : \dot{y}\dot{t}^2 :: y \times \overline{\ddot{t}\dot{x} - \dot{t}\ddot{x}} : \dot{y}^2\dot{t}$.

Corol. 3. In the same manner, $\frac{\dot{t}^2}{\dot{y}^2} = 1 + \frac{\dot{x}^2}{\dot{y}^2} = 1 + \frac{1}{z^2}$ and $\frac{\dot{t}\ddot{t}\dot{y} - \dot{t}^2\ddot{y}}{\dot{y}^3} = \frac{\dot{z}}{z^3} = \frac{\dot{z}\dot{x}^3}{\dot{y}^3}$, therefore $\frac{\dot{t}\ddot{t}\dot{y} - \dot{t}^2\ddot{y}}{\dot{x}} \times y = y\dot{x}^2\dot{z}$ and TK : Pp :: $\frac{\dot{t}\ddot{t}\dot{y} - \dot{t}^2\ddot{y}}{\dot{x}} \times y : \dot{y}\dot{t}^2 :: \overline{y \times \ddot{t}\dot{y} - \dot{t}\ddot{y}} : \dot{x}\dot{y}\dot{t}$.

Corol. 4. Since $z = \frac{\dot{y}}{\dot{x}}$ we have $\dot{z} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2}$ and TK : Pp :: $y \times \overline{\ddot{y}\dot{x} - \dot{y}\ddot{x}} : \dot{y}\dot{t}^2$.

Prop. XXX.

152. *To find the radius of curvature at any point of a given curve.*

Supposing all things as in the last article, let C be the center of curvature, and CP the radius; now since the radius of curvature is perpendicular to the curve, the angles CPT, Cpt will be right ones, and Cp will be parallel to TK, and the angle KTp will be equal to the angle PpC, therefore the triangles

triangles CPp, pKT will be similiar and $CP : pK :: Pp : TK$, whence $CP = \frac{pK \times pP}{TK} = \frac{PT \times Pp}{TK}$ $= \frac{y\dot{t}}{\dot{y}} \times \frac{Pp}{TK}$, but by Lemma V. $\frac{Pp}{TK} = -\frac{\dot{y}\dot{t}^2}{y\dot{x}^2\dot{z}}$ whence $CP = \frac{-y\dot{t}}{\dot{y}} \times \frac{\dot{y}\dot{t}^2}{y\dot{x}^2\dot{z}} = \frac{-\dot{t}^3}{\dot{x}^2\dot{z}}$.

We may find some different expressions for CP, by the different expressions for $\frac{Pp}{TK}$ in the corollaries to Lemma V. Thus by Corol. 2. $CP = \frac{y\dot{t}}{\dot{y}} \times \frac{\dot{y}\dot{t}}{y \times \ddot{t}\dot{x} - \dot{t}\ddot{x}} = \frac{\dot{y}\dot{t}^2}{\ddot{t}\dot{x} - \dot{t}\ddot{x}}$. By Corol. 3. $CP = \frac{y\dot{t}}{\dot{y}} \times \frac{\dot{x}\dot{y}\dot{t}}{y \times \ddot{t}\dot{y} - \dot{t}\ddot{y}} = \frac{\dot{x}\dot{t}^2}{\ddot{t}\dot{y} - \dot{t}\ddot{y}}$. And by Corol. 4. $CP = \frac{y\dot{t}}{\dot{y}} \times \frac{\dot{y}\dot{t}^2}{y \times \ddot{x}\dot{y} - \dot{x}\ddot{y}} = \frac{\dot{t}^3}{\ddot{x}\dot{y} - \dot{x}\ddot{y}}$. And by computing any of these expressions we may find the radius of curvature, but we shall make use of that in the next corollary as being the simplest.

Corol. 1. Since $\dot{t}^3 : \dot{x}^3 :: PT^3 : TO^3 :: DP^3 : OP^3$ we shall have $CP = \frac{-\dot{t}^3}{\dot{x}^2\dot{z}} = \frac{-\dot{t}^3\dot{x}}{\dot{x}^3\dot{z}} = \frac{-DP^3}{OP^3} \times \frac{\dot{x}}{\dot{z}}$.

Corol. 2. Let the ordinate OP meet the circle of curvature in V, then fV is parallel to AO, and $PD : PO :: Pf : PV$ whence $PV = \frac{Pf \times PO}{PD} =$ $\frac{2CP \times PO}{PD} = 2 \times \frac{DP^2}{OP^2} \times \frac{\dot{x}}{\dot{z}} = \frac{-2\dot{t}^2}{\dot{x}\dot{z}}$ and if CE is

drawn

drawn parallel to AO, PE $= \frac{1}{2}$ PV $= \frac{-\dot{t}^2}{\dot{x}\dot{z}} = \frac{DP^2}{OP^2} \times \frac{\dot{x}}{\dot{z}}$.

Corol. 3. In the same manner $fV = \frac{Pf \times OD}{PD} = \frac{2DP^3 \times OD}{OP^3 \times PD} \times \frac{\dot{x}}{\dot{z}} = \frac{2DP^2 \times OD}{OP^3} \times \frac{\dot{x}}{\dot{z}}$. But $\frac{OD}{OP} = \frac{\dot{y}}{\dot{x}} = z$, therefore $fV = \frac{2\dot{t}^2 z}{\dot{x}\dot{z}}$ and CE $= \frac{-\dot{t}^2 z}{\dot{x}\dot{z}} = \frac{DP^2}{OP^2} \times \frac{z\dot{x}}{\dot{z}}$.

Corol. 4. Let the curve AP (fig. 41.) be perpendicular to the base AB in the point A, then will CP $=$ DP $= \frac{y\dot{y}}{\dot{x}} = yz$. And if the curve touches AL in A then the radius of curvature in the point A $= \frac{x\dot{x}}{\dot{y}} = \frac{x}{z}$. AL being taken for the abscissa.

153. Let AK (fig. 42.) be the radius of curvature at the point A, and let the locus of the point C be the curve KC draw CM perpendicular to the abscissa AO, then AM $=$ AO $+$ OM $=$ AO $+$ EC $= x - \frac{\dot{t}^2 z}{\dot{x}\dot{z}}$ we have also CM $=$ OE $=$ PE $-$ PO $= - \frac{\dot{t}^2}{\dot{x}\dot{z}} - y$. Call AK, g, KM, s, MC, v, put $\frac{\dot{t}^2}{\dot{x}\dot{z}} = w$ then $s = x - wz - g$, and $v = -w - y$, whence $\frac{s + g - x}{z} = -w = v + y$ and $s + g$

$- x$

$-x = vz + yz$, therefore by the help of the equation between x and y, we may find the relation between s and v, or the equation to the curve KC.

PROP. XXXI.

154. *Let* PC *be the radius of curvature in the point* P *and* KC *the locus of the point* C. *I say* PC *touches the curve* KC *in the point* C, *and the arc* KC *of the curve* KC *is equal to the difference between* PC *and* AK.

Because $OD = \frac{y\dot{y}}{\dot{x}} = yz$ we have $DM = OM - OD = -wz - yz =$ (by art. 153.) vz.

Since $s = x - wz - g$ we have $\dot{s} = \dot{x} - w\dot{z} - \dot{w}z$, but $w = \frac{\dot{t}^2}{\dot{x}\dot{z}}$, therefore $w\dot{z} = \frac{\dot{t}^2}{\dot{x}}$ and $\dot{s} = \dot{x} - \frac{\dot{t}^2}{\dot{x}} - \dot{w}z = -\frac{\dot{y}^2}{\dot{x}} - \dot{w}z = -\dot{y}z - \dot{w}z$, and since $v = -w - y$ we have $\dot{v} = -\dot{w} - \dot{y}$ and $\frac{\dot{s}}{\dot{v}} = z$, therefore the subtangent $\frac{v\dot{s}}{\dot{v}} = vz$, but we have proved that DM is also $= vz$ therefore DM is the subtangent and consequently CD the tangent. Q. E. D. 1°.

Call CP, r, and put $\frac{\dot{t}}{\dot{x}} = Z$ then $r = \frac{-\dot{t}^3}{\dot{x}\dot{z}} = -wZ$ and $\dot{r} = -\dot{w}Z - w\dot{Z}$. But since $\frac{\dot{t}^2}{\dot{x}^2} = 1 + \frac{\dot{y}^2}{\dot{x}^2}$ or $Z^2 = 1 + z^2$ we have $Z\dot{Z} = z\dot{z}$ and $w\dot{Z} =$

wzż

$\frac{wz\dot{z}}{Z} = \frac{\dot{t}^2 z}{\dot{x}Z} = \frac{\dot{t}^2}{\dot{x}^2} \times \frac{z\dot{x}Z}{Z^2} = z\dot{x}Z = \dot{y}Z$. Therefore $\dot{r} = -\dot{w}Z - \dot{y}Z = \dot{v}Z$.

Since by the former part of this proposition $\frac{\dot{s}}{\dot{v}} = z$, we have $\dot{s}^2 + \dot{v}^2 = \dot{v}^2 \times \overline{1+z^2} = \dot{v}^2Z^2$, and the fluxion of the arc KC $= \sqrt{\dot{s}^2 + \dot{v}^2} = \dot{v}Z = \dot{r}$, therefore taking the fluents KC $= r + C$ which is to vanish when $r = g$ therefore, $C = -g$ and KC $= r - g$. Q. E. D. 2°.

155. *Corol.* Hence if a flexible line or thread in length AK + KC or PC be applied round the curve KC from C to K, and let the part KA remain a strait line; then by unwinding the thread so that the part, which is not applied round the curve, may be extended in a right line, the end A of this thread will describe the curve AP.

The curve AP described in this manner is said to be described by the evolution of the curve KC. The curve KC is called the *Evoluta* and the curve AP the *Involuta*.

156. The variation of curvature, or rather the variation of the radius of curvature, is how much the radius is varied in its progress through different parts of the curve.

PROP.

Prop. XXXII.

157. *The variation of curvature is as the fluxion of the radius directly, and as the fluxion of the curve inversely.*

When the moment of the arc is given, it is plain that the variation is as the moment of the radius of curvature. If the moment of the radius is given, let AB, DE (fig. 44.) be two unequal infinitely small arcs, and take the arc DG equal to the arc AB, then it is plain that the variation through the arc DE is the greater, the less the arc DE is, and the contrary, therefore the variation in DG is to the variation in DE as DE : DG; but the variation in AB is equal to the variation in DG because AB $=$ DG, therefore the variation in AB is to the variation in DE as DE : AB or inversely as the moments of the arcs. Therefore when neither the moment of the arc, nor the moment of the radius is given, the variation will be as the moment of the radius directly and the moment of the arc inversely, or as the fluxion of the radius directly and as the fluxion of the arc inversely. Q. E. D.

158. *Corol.* Hence the variation of curvature is as the difference between the fluxion of half the chord of the circle of curvature perpendicular to the base, and the fluxion of the ordinate directly and as the fluxion of the abscissa inversely. For supposing the same things as in Prop. XXXI. $\dot{r} = \dot{v}Z = \frac{\dot{v}t}{\dot{x}}$ whence the variation or $\frac{\dot{r}}{\dot{t}}$ is $= \frac{\dot{v}}{\dot{x}}$ $= \frac{-\dot{w}-\dot{y}}{\dot{x}}$ but $-\ w = \text{PE} = \frac{1}{2}$ the chord. Therefore the variation is the fluxion of PE$-$OP or OE directly and the fluxion of AO inversely.

Examples

EXAMPLES *of finding the Curvature and its Variation.*

Example I. *Fig.* 7.

159. *To find the radius of Curvature in an hyperbola.*

The equation to the hyperbola is (art. 100.) $ax+bx^2=y^2$, a being the latus rectum, and $\frac{a}{b}$ the tranverse axis. Divide by x^2 then $\frac{a}{x}+b=\frac{y^2}{x^2}$, and taking the fluxions $\frac{-a\dot{x}}{x^2}=\frac{2y\dot{y}x-2y^2\dot{x}}{x^3}$ or $-ax\dot{x}=2y\dot{y}x-2y^2\dot{x}$ and $\frac{\dot{y}}{\dot{x}}=z=\frac{2y^2-ax}{2xy}=\frac{y}{x}-\frac{a}{2y}$, taking the fluxions again, we have $\dot{z}=\frac{\dot{y}}{x}-\frac{y\dot{x}}{x^2}+\frac{a\dot{y}}{2y^2}=\frac{2y^2+ax}{2xy^2}\times\dot{y}-\frac{y\dot{x}}{x^2}$ and $\frac{\dot{z}}{\dot{x}}=\frac{2y^2+ax}{2xy^2}\times\frac{\dot{y}}{\dot{x}}-\frac{y}{x^2}$. But $\frac{\dot{y}}{\dot{x}}=\frac{2y^2-ax}{2xy}$ therefore $\frac{\dot{z}}{\dot{x}}=\frac{4y^4-a^2x^2}{4x^2y^3}-\frac{y}{x^2}=\frac{y}{x^2}-\frac{a^2}{4y^3}-\frac{y}{x^2}=\frac{-a^2}{4y^3}$. And the radius of curvature CP =

DP³

$\frac{DP^3}{y^3} \times \frac{4y^3}{a^2} = \frac{4 \times DP^3}{a^2} =$ (by art. 109) $\frac{4}{a^2} \times \overline{\frac{a^2}{4} + y^2 \times \overline{1+b}}\big|^{\frac{3}{2}}$.

Hence $PE = \frac{DP^2}{y^2} \times \frac{\dot{x}}{\dot{z}} = \frac{DP^2 \times 4y}{a^2} = y + \frac{4y^3}{a^2} \times \overline{1+b} = -w$, and $PE - OP = -w - y = v = \frac{4y^3}{a^2} \times \overline{1+b}$, and taking the fluxions $\dot{v} = \frac{12y^2\dot{y}}{a^2} \times \overline{1+b}$, and $\frac{\dot{v}}{\dot{x}} = \frac{12y}{a^2} \times \overline{1+b} \times \frac{y\dot{y}}{\dot{x}} = \frac{12y \times \overline{1+b}}{a^2} \times OD$, therefore the variation of curvature is as $\frac{12y \times \overline{1+b}}{a^2} \times OD$ or $OP \times OD \times \frac{12 \times \overline{1+b}}{a^2}$.

Since $z = \frac{y}{x} - \frac{a}{2y}$ we have $yz = \frac{y^2}{x} - \frac{a}{2} =$ (when $y = 0$) $\frac{1}{2} a =$ the radius of curvature in the point A (Prop. XXX. Cor. 4.)

160. If the curve be an ellipsis, (fig. 6.) then b will be negative and the radius $CP = \frac{4 \times DP^3}{a^2} = \frac{4}{a^2} \times \overline{\frac{a^2}{4} + y^2 \times \overline{1-b}}\big|^{\frac{3}{2}}$. $PE = y + \frac{4y^3}{a^2} \times \overline{1-b}$. And the variation of curvature will be as $OP \times OD \times \frac{12 \times \overline{1-b}}{a^2}$.

161. If

161. If the curve is a parabola (fig. 5.) then $b = 0$, and the radius of curvature CP $= \frac{4 \times DP^3}{a^2} = \frac{4}{a^2} \times \overline{\frac{a^2}{4} + y^2}|^{\frac{3}{2}}$. PE $= y + \frac{4y^3}{a^2}$, and the variation of curvature will be as OP $\times$ OD $\times \frac{12}{a^2} =$ (because OD $= \frac{1}{2} a$. art. 109.) $\frac{6}{a} \times$ OP. The radius of curvature at the vertex A is $\frac{a}{2} = g$ and the equation to the evoluta is (art. 153.) $s + \frac{a}{2} - x = \overline{v + y} \times z$. But $x = \frac{y^2}{a}$, and $z = \frac{a}{2y}$, which being substituted in the equation, we have $s + \frac{a}{2} - \frac{y^2}{a} = \frac{av}{2y} + \frac{a}{2}$ or $s = \frac{y^2}{a} + \frac{av}{2y}$ but $v = \frac{4y^3}{a^2}$ or $\frac{a^2 v}{4} = y^3$ therefore, $s = \frac{av}{4y} + \frac{av}{2y} = \frac{3av}{4y}$ and cubing both sides $s^3 = \frac{27a^3v^3}{64y^3} = \frac{27}{16} av^2$ which shews the evoluta to be a parabola of the second kind whose parameter is $\frac{27}{16} a$.

162. Let APB (fig. 6.) be an ellipse or hyperbola, S its center, draw the semidiameter SN the conjugate to SP meeting the perpendicular to the curve PD in F, then the radius of curvature CP $= \frac{SN^2}{PF}$, for by conic sections SN $\times$ PF $=$ SG $\times$ SB and DP $\times$ PF $=$ SG^2 whence DP$^2 =$

SG^4

$\frac{SG^2}{PF} = \frac{SG \times SN}{SB}$ and $DP^3 = \frac{SG^3 \times SN^3}{SB^3}$. Therefore since $\frac{a}{2} = \frac{SG^2}{SB}$ we have the radius of curvature $CP = \frac{4}{a^2} \times DP^3 = \frac{SB^2}{SG^4} \times \frac{SG^3 \times SN^3}{SB^3} =$

$\frac{SN^3}{SG \times SB} = \frac{SN^3}{SN \times PF} = \frac{SN^2}{PF}$.

163. The chord of the circle of curvature that passes through the center S (fig. 48.) is equal to the parameter of the diameter SP, for let Pb be that chord, and from the center of curvature C let fall the perpendicular CH on Pb then PH is half that chord, and the triangles PSF, PCH, being similar, we have PS : PF :: PC : PH and PH $=$ $\frac{PF \times PC}{PS} = \frac{SN^2}{SP}$ and Pb $= \frac{2SN^2}{SP} =$ the parameter of the diameter SP.

164. Let Pv be any other chord, which cuts SN in L, join vb, and since the right line TP touches the circle, the angle Pvb $=$ angle TPH $=$ angle PSL, therefore the triangles PSL, Pvb are similar, and PL : PS :: Pb : Pv whence Pv $=$ $\frac{PS \times Pb}{PL} = \frac{2SN^2}{PL}$.

If the chord Pv passes through the focus then PL $=$ AS $= \frac{1}{2}$ AB and the chord Pv $= \frac{4SN^2}{AB} =$ $\frac{Nn^2}{AB}$.

Let the chord Pv meet the conic section in Z, draw the diameter Sg parallel to PZ, meeting the

tangent PT in t, draw $rS\varrho$ the conjugate to Sg meeting PZ in l, then by conic sections $Sg^2 = PL \times Pl$, and $Pl = \frac{Sg^2}{PL}$ and $PZ = 2Pl = \frac{2Sg^2}{PL}$ therefore $Pv : PZ :: \frac{2SN^2}{PL} : \frac{2Sg^2}{PL} :: SN^2 : Sg^2$.

Let the right line ϱT touch the curve in ϱ, and let it meet PT in T, then $PT^2 : \varrho T^2 :: SN^2 : Sg^2 :: Pv : PZ$.

Let the curve and circle meet in v then $Pv = PZ$ and $PT = \varrho T$, and $SP = S\varrho$; therefore the angle $PTS = \varrho TS = TmP$ and the angle $OPT = OPv$.

165. Let Q (fig. 46.) be the focus, and QY perpendicular to the tangent, then by art. 110. Corol. 3. $PD = \frac{\frac{1}{2}a \times QP}{QY}$ and $PD^3 = \frac{\frac{1}{8}a^3 \times QP^3}{QY^3}$ whence $CP = \frac{4}{a^2} \times PD^3 = \frac{\frac{1}{2}a \times QP^3}{QY^3}$. On the right line QP let fall the perpendicular Ce from C then $QP : QY :: CP : Pe$ and $Pe = \frac{QY \times CP}{QP} = \frac{\frac{1}{2}a \times QP^2}{QY^2}$, and the chord passing through the focus $= 2Pe = \frac{a \times QP^2}{QY^2}$.

166. Let AP (fig. 49.) be a parabola, Q the focus, then $CP = \frac{QP \times PD}{AQ}$. For $PD^2 = TD \times OD = 2TQ \times 2AQ = a \times QP$ therefore $PD^3 = a \times QP \times PD$ and $CP = \frac{4 \times PD^3}{a^2} = \frac{4 \times QP \times PD}{a} = \frac{QP \times PD}{AQ}$.

Let

Let Pb be the chord parallel to the axis AO, draw CH parallel to PO, then PD : PC :: DO : EC and EC or PH $= \frac{PC \times DO}{PD} = \frac{QP \times DO}{AQ} =$ 2QP and the chord Pb = 4QP = the parameter of the diameter PH.

Let Pv be any other chord meeting the axis in L, and the parabola in Z, join vb, the triangles TPL, vbP, are similiar, therefore PL : TL :: Pb : Pv and P$v = \frac{TL \times Pb}{PL} = \frac{4QP \times TL}{PL}$.

If L be the focus, then TL = QP = PL and the chord = Pb, therefore the chord passing thro' the focus is equal to the chord parallel to the axis, that is, equal to the parameter of the diameter PH = 4QP.

Bisect the right line PZ in l, and through l draw the diameter $l\varrho$ meeting the tangent PT in t, then since P$v = \frac{4QP \times TL}{PL} = \frac{4QP \times tl}{Pl}$, we have P$v$: Pl :: 4QP $\times$ tl : Pl^2 :: 4QP $\times$ tl : 4Qϱ $\times$ ϱl :: 2QP $\times$ tl : Qϱ $\times$ tl :: 2QP : Qϱ, whence Pv : 2Pl or PZ :: QP : Qϱ.

Draw $\varrho\tau$ touching the parabola in ϱ, and meeting AO in τ and let the circle and parabola intersect each other in v, then PZ = Pv, and QP = Qϱ, therefore $\varrho\tau$ = PT and the points T, τ will coincide, and the angle OTP = OTϱ = OLP and the angle OPT = OPL.

Example II. *Fig.* 9.

167. *To find the radius of curvature in the Cissoid of Diocles.*

Call AB, a; AO, x; OP, y; then by art. 111. $z = \frac{\dot{y}}{\dot{x}} = \frac{3x^2y + y^3}{2x^3} = \frac{3y}{2x} + \frac{y^3}{2x^3}$ and taking the fluxions $\dot{z} = \frac{3\dot{y}}{2x} - \frac{3y\dot{x}}{2x^2} + \frac{3y^2\dot{y}}{2x^3} - \frac{3y^3\dot{x}}{2x^4} = \frac{3\dot{y}}{2} \times \frac{x^2 + y^2}{x^3} - \frac{3y\dot{x}}{2} \times \frac{x^2 + y^2}{x^4}$ and $\frac{\dot{z}}{\dot{x}} = \frac{3}{2} \times \frac{x^2 + y^2}{x^3} \times \overline{\frac{\dot{y}}{\dot{x}} - \frac{y}{x}}$ but $\frac{\dot{y}}{\dot{x}} = \frac{3y}{2x} + \frac{y^3}{2x^3}$ and $\frac{\dot{y}}{\dot{x}} - \frac{y}{x} = \frac{y}{2} \times \frac{x^2 + y^2}{x^3}$ therefore $\frac{\dot{z}}{\dot{x}} = \frac{3y}{4} \times \frac{\overline{x^2 + y^2}|^2}{x^6}$. But $x^6 = y^4 \times \overline{a - x}|^2$ therefore $\frac{\dot{z}}{\dot{x}} = \frac{3y}{4} \times \frac{\overline{x^2 + y^2}|^2}{y^4 \times \overline{a - x}|^2} = \frac{3}{4} \times \frac{\overline{x^2 + y^2}|^2}{y^3 \times \overline{a - x}|^2}$ and CP $= \frac{DP^3}{y^3} \times \frac{\dot{x}}{\dot{z}} = \frac{4}{3} \times \frac{DP^3 \times \overline{a - x}|^2}{\overline{x^2 + y^2}|^2} = \frac{4DP^3 \times BO^2}{3AP^4}$.

The curve touches the right line AB in A, because when $x = 0$, $\frac{OD}{y} = z = \frac{3y}{2x} + \frac{y^3}{2x^3} = 0$. Therefore the radius of curvature in A is equal $\frac{x}{z}$ (art. 152. Cor. 4) $= \frac{2x^4}{3x^3y + y^3} = 0$. (Since at the

the vertex A. $y^2 = \frac{x^3}{a}$) therefore the radius of curvature at A is $= 0$, and the curvature is infinite or greater than that of any circle.

Example III. *Fig.* 8.

168. *To find the radius of curvature in the* Conchoid *of* Nicomedes.

Call AG $=$ PR, a; FG, b; GO, x; OP, y; RQ $= \sqrt{a^2 - x^2} = v$ and put $a^2 b + x^3 = q$, then by Corol. art. 112. we have $z = \frac{\dot{y}}{\dot{x}} = \frac{q}{x^2 v}$ and taking the fluxions $\dot{z} = \frac{\dot{q}}{x^2 v} - \frac{2q\dot{x}}{x^3 v} - \frac{q\dot{v}}{x^2 v^2}$

But $\dot{q} = 3x^2\dot{x}$ and $\dot{v} = \frac{-x\dot{x}}{v}$ therefore $\frac{\dot{z}}{\dot{x}} = \frac{3}{v} - \frac{2q}{x^3 v} + \frac{q}{xv^3} = \frac{3x^3v^2 - 2qv^2 + qx^2}{x^3 v^3}$. But since $q = a^2 b + x^3$ we have $3x^3 - 2q = 3q - 3a^2 b - 2q = q - 3a^2 b$ therefore $\frac{\dot{z}}{\dot{x}} = \frac{qv^2 - 3a^2bv^2 + qx^2}{x^3 v^3}$

$$= \frac{qa^2 - 3a^2bv^2}{x^3v^3} = \frac{a^2 \times \overline{q - 3bv^2}}{x^3 v^3} = \frac{a^2 \times \overline{\frac{qy}{x^2 v} - \frac{3bvy}{x^2}}}{yxv^2}.$$

Now $\frac{qy}{x^2 v} = yz =$ OD, and as the triangles PQR, FGR, FOP are similiar, PQ : QR :: FG : RG and RG $= \frac{bv}{x}$ and $\frac{bvy}{x^2} = \frac{\text{RG} \times \text{OP}}{\text{PQ}}$, also PR :

QR :: FP : OP and $v^2 = \frac{a^2y^2}{FP^2}$ therefore substituting these values we have $\frac{\dot{z}}{\dot{x}} =$

$$\frac{\overline{a^2 \times FP^2 \times OD - \frac{3RG \times OP}{PQ}}}{xa^2y^3} = \frac{\overline{FP^2 \times OD - \frac{3RG \times OP}{PQ}}}{PQ \times OP^3}$$

and the radius of curvature $CP = \frac{DP^3}{OP^3} \times \frac{\dot{x}}{\dot{z}} =$

$$\frac{PQ \times DP^3}{FP^2 \times OD \times PQ - 3RG \times OP},$$

Since $z = \frac{q}{x^2v}$ we have $yz = \frac{qy}{x^2v} = \frac{q \times \overline{b+x}}{x^3}$, but if $y = 0$ then $x = \pm a$ and the radius of curvature in A or $a = \frac{\overline{a^2 b \pm a^3} \times \overline{b \pm a}}{a^3} = \frac{\overline{b \pm a}|^2}{a}$ therefore as GA : FA :: FA : radius of curvature in A, and Ga : Fa :: Fa : radius of curvature in a.

Example

Example IV. *Fig.* 5. 13.

169. *To find the radius of curvature in a parabola or hyperbola whose equation is* $y = ax^m$.

By this equation we have $z = \frac{\dot y}{\dot x} = max^{m-1}$, and $\dot z = m \,.\, \overline{m-1} \,.\, ax^{m-2}\,\dot x = m.\overline{m-1} \times \frac{y\dot x}{x^2}$ therefore $\frac{\dot x}{\dot z} = \frac{xx}{m \,.\, \overline{m-1} \,.\, y}$ and the radius of curvature $CP = \frac{DP^3}{OP^3} \times \frac{\dot x}{\dot z} = \frac{DP^3 \times x^2}{m \,.\, \overline{m-1} \,.\, y^4}$.

But $DO = \frac{my^2}{x}$ and $DO^2 = \frac{m^2y^4}{x^2}$; whence $\frac{x^2}{y^4} = \frac{m^2}{DO^2}$ and $CP = \frac{m \times DP^3}{\overline{m-1} \times DO^2}$.

Since $DO = \frac{my^2}{x}$ we have $y^2 = \frac{DO \times x}{m}$ and $DP^2 = DO^2 + y^2 = DO \times \overline{DO + \frac{x}{m}}$ therefore $\frac{DP^2}{DO^2} = \frac{m \times DO + x}{m \times DO}$ and $CP = \frac{DP \times \overline{m \times DO + x}}{\overline{m-1} \times DO}$ whence $\overline{m-1} \times DO : m \times DO + x :: DP : CP$.

Also $PE = \frac{PC \times PO}{PD} = \frac{PO \times \overline{m \times DO + x}}{\overline{m-1} \times DO}$ and $CE = \frac{PC \times DO}{PD} = \frac{m \times DO + x}{m-1}$.

Since $DP^2 = OD^2 + OP^2 = \frac{m^2y^4}{x^2} + y^2$ we shall have $\frac{DP^2}{OP^2} = \frac{\dot t^2}{\dot x^2} = \frac{m^2y^2}{x^2} + 1$ and $w = \frac{\dot t^2}{\dot x^2} \times \frac{\dot x}{\ddot z} = \frac{m^2y^2x^2}{m \,.\, \overline{m-1} \,.\, yx^2} + \frac{x^2}{m \,.\, \overline{m-1} \,.\, y} = \frac{my}{m-1} + \frac{\dot x^2}{m \,.\, \overline{m-1} \,.\, y}$ whence $w + y = \frac{\overline{2m-1} \times y}{m-1} + \frac{x^2}{m \,.\, \overline{m-1} \,.\, y} = \frac{\overline{2m-1} \times ax^m}{m-1} + \frac{x^{4-m+2}}{m \,.\, \overline{m-1} \,.\, a}$ and taking the fluxions $\frac{\dot w + \dot y}{\dot x} = \frac{m \,.\, \overline{2m-1} \,.\, ax^{m-1}}{m-1} + \frac{\overline{2-m} \,.\, x^{-m+1}}{m \,.\, \overline{m-1} \,.\, a} = \frac{m \,.\, \overline{2m-1} \,.\, y}{\overline{m-1} \,.\, x} + \frac{\overline{2-m} \,.\, x}{m \,.\, \overline{m-1} \,.\, y}$ and the variation of curvature $= \frac{-\dot w - \dot y}{\dot x} = \frac{m \,.\, \overline{1-2m} \,.\, y}{\overline{m-1} \,.\, x} + \frac{\overline{m-2} \,.\, x}{m \,.\, \overline{m-1} \,.\, y}$.

In the common parabola $y = ax^{\frac{1}{2}}$ (a^2 being the latus rectum) and $m = \frac{1}{2}$, the radius of curvature $CP = \frac{DP \times \overline{\frac{1}{2}DO + AO}}{-\frac{1}{2} \times DO}$. But if Q is the focus, $\frac{1}{2}DO = AQ$ and $CP = \frac{DP \times \overline{AQ + AO}}{AQ} = \frac{DP \times TQ}{AQ}$ as we found in art. 166. $PE = \frac{PO \times TQ}{AQ}$ and $CE = 2TQ$. The variation of curvature is as $\frac{\overline{\frac{1}{2}-2} \,.\, x}{\frac{1}{2} \,.\, -\frac{1}{2}y} = \frac{6x}{y} = \frac{6y}{a^2}$.

In

In the equilateral hyperbola (fig. 13.) $xy = a$; and $m = -1$, and the radius of curvature $CP = \frac{DP \times \overline{-DO + AO}}{-2DO} =$ (because $-DO = \frac{y^2}{x}$) $\frac{DP \times \frac{y^2 + x^2}{x}}{\frac{2y^2}{x}} = \frac{DP \times AP^2}{2OP^2}$ therefore $2OP^2 : AP^2 :: DP : PC$ or since $AP = PT$, $CP = \frac{DP \times PT^2}{2OP^2} = \frac{DP^3}{2DO^2}$, $PE = \frac{PO \times PT^2}{2OP^2} = \frac{PT^2}{2OP}$ and $CE = \frac{DP^2}{2DO}$. The variation of curvature is as $\frac{3y}{2x} - \frac{3x}{2y} = \frac{3}{2} \times \frac{y^2 - x^2}{xy} = \frac{3}{2} \times \frac{y^2 - x^2}{a}$.

170. If the curve AP (fig. 50.) is a parabola, and m is less than 1, it will cut the base at right angles in A, by art. 113. therefore by Prop. XXX. Corol. 4 the radius in A is equal to $yz = max^{m-1}y = \frac{my^2}{x} = ma^2x^{2m-1}$. If $m = \frac{1}{2}$ the curve will be the Apollonian parabola, and the radius $= \frac{y^2}{2x} =$ half the latus rectum.

If m is greater than $\frac{1}{2}$, or if $2m - 1$ is affirmative, the radius will be infinitely small, and the curvature infinitely great or greater than that of any circle.

If m is less than $\frac{1}{2}$, then $2m - 1$ is negative and the radius of curvature will be infinite, and the curvature

curvature infinitely small or less than that of any circle.

Let AΠ be another parabola in which OΠ is as x^{m+n} or whose equation putting O$\Pi = v$ is $v = bx^{m+n}$, call the radius of curvature of the curve AP at the point A, r, that of the curve AΠ, R, then $R = \frac{\overline{m+n} \times v^2}{x}$ and $R : r ::$ $\overline{m+n} \times v^2 : my^2 :: \overline{m+n} \times b^2x^{2m+2n} : ma^2x^{2m} :: x^{2n} : \frac{ma^2}{\overline{m+n} \times b^2}$ that is as o to the finite quantity $\frac{ma^2}{\overline{m+n} \times b^2}$ therefore R is infinitely less than r, and the curvature of AΠ is infinitely greater than that of AP.

Corol. Hence it appears that there are serieses of curves each of which has the curvature at the vertex infinitely greater or infinitely less than the preceding *.

Example V. *Fig.* 15.

171. ***To find the radius of curvature in the logarithmic curve.***

Call AO, x; OP, y; OT, a; then $z = \frac{\dot{y}}{\dot{x}} = \frac{y}{a}$ (art. 115.) and $\dot{z} = \frac{\dot{y}}{a} = \frac{\dot{x}y}{a^2}$ whence $\frac{\dot{x}}{\dot{z}} = \frac{a^2}{y}$ and the radius of curvature $CP = \frac{DP^3}{OP^3} \times \frac{\dot{x}}{\dot{z}} = \frac{DP^3 \times OT^2}{OP^4}$. But $\frac{DP^3}{OP^3} = \frac{PT^3}{OT^3}$, therefore CP=

* Newton Princip. Lib. I. Sect. I. schol.

PT³

$\frac{PT^3 \times OT^2}{OT^3 \times OP} = \frac{PT^3}{OP \times OT}$. And PE $=$ $\frac{CP \times PO}{PD} = \frac{CP \times OT}{PT} = \frac{PT^2}{OP}$.

Since $DP^2 = DO^2 + OP^2 = \frac{y^4}{a^2} + y^2$ we have $\frac{DP^2}{OP^2} = \frac{\dot{t}^2}{\dot{x}^2} = \frac{a^2+y^2}{a^2}$. But $\frac{\dot{x}}{\dot{z}} = \frac{a^2}{y}$ therefore $\frac{\dot{t}^2}{\dot{x}\dot{z}}$ or $w = \frac{a^2+y^2}{y}$ and $w + y = \frac{a^2}{y} + 2y$ and taking the fluxions $\dot{w} + \dot{y} = \frac{-a^2\dot{y}}{y^2} + 2\dot{y} =$ $\frac{2y^2 - a^2}{y^2} \times \dot{y} = \frac{2y^2 - a^2}{y^2} \times \frac{y\dot{x}}{a}$ and $\frac{\dot{w}+\dot{y}}{\dot{x}} =$ $\frac{2y^2 - a^2}{ay} = \frac{\frac{2y^2}{a} - a}{y} = \frac{2DO - OT}{OP}$ therefore the variation of curvature $\frac{-\dot{w}-\dot{y}}{\dot{x}}$ is as $\frac{OT - 2DO}{OP}$.

Example VI. *Fig.* 17.

172. *To find the radius of Curvature in a Cycloid.*

Call CH, a, CA, b, CO, x, OP, y, OM, v, then by art. 117. $\frac{\dot{y}}{\dot{x}} = z = \frac{b+x}{v}$ and taking the fluxions $\dot{z} = \frac{\dot{x}}{v} - \frac{b+x}{v^2} \times \dot{v}$, but $-\dot{v} = \frac{x\dot{x}}{v}$ therefore

therefore $\frac{\dot{z}}{\dot{x}} = \frac{1}{v} + \frac{\overline{b+x} \times x}{v^3} = \frac{v^2 + bx + x^2}{v^3} = \frac{a^2 + bx}{v^3}$. Also since the triangles AOm, DOP are similar we have $\frac{DP^3}{OP^3} = \frac{Am^3}{Om^3} = \frac{Am^3}{v^3}$ therefore the radius of curvature $cP = \frac{DP^3}{OP^3} \times \frac{\dot{x}}{\dot{z}} = \frac{Am^3}{a^2 + bx} = \frac{Am^3}{CH^2 + CA \times CO}$.

Let V be the vertex then the abscissa CO$=$CK and Am $=$AK and the radius of curvature in V$=$ $\frac{AK^3}{CK^2 + CA \times CK} = \frac{AK^3}{CK \times AK} = \frac{AK^2}{CK}$.

173. In the common cycloid (fig. 18.) $a=b$, $Am^2 = AO \times AL = 2AC \times AO$, and $CH^2 + CA \times CO = CA \times AO$ therefore $cP = 2Am$, $cE = 2Om$, and $EP = 2AO$.

Let VB be the axis, produce it downwards to a, so that $Ba = BV = AL$, bisect Ba in $\varkappa$ and through c draw $c\omega$ parallel to AB, and about the center $\varkappa$ with the radius $\varkappa B = CA$ describe the circle $B\mu a$, then since $\omega B + AO = EP = 2AO$ we have $\omega B = AO$ and $\omega\mu = Om$ and the arc $Lm = a\mu$. Let ac be the evoluta, then $\omega E = \omega c + cE = \omega c + 2mO$, and $AB - \omega c - 2mO = AB - \omega E = OP =$ arc $Am - mO$, therefore $\omega c + mO = AB - Am = Lm$ or $\omega c + \omega\mu =$ arc $a\mu$, whence it appears that that the evoluta cA is a semi-cycloid, equal to the semi-cycloid APV, and whose axis is $Al = AL$ and generating circle $a\mu B$.

Therefore the arc of the cycloid $Ac = cP = 2Am = 2B\mu$. Since

Since $\frac{\dot{t}^2}{\dot{x}^2} = \frac{DP^2}{OP^2} = \frac{Am^2}{Om^2} = \frac{AL}{OL} = \frac{2a}{a-x}$ $= \frac{2a \times \overline{a+x}}{v^2}$ and $\frac{\dot{x}}{\dot{z}} = \frac{v^2}{a \times a+x}$ we have $w = \frac{\dot{t}^2}{\dot{x}^2}$ $\times \frac{\dot{x}}{\dot{z}} = 2v$ and $w + y = 2v + y$ and taking the fluxions $\dot{w} + \dot{y} = 2\dot{v} + \dot{y} = \frac{-2x\dot{x}}{v} +$ $\frac{a\dot{x} + x\dot{x}}{v} = \frac{a\dot{x} - x\dot{x}}{v}$ and the variation of curvature $\frac{-\dot{w} - \dot{y}}{\dot{x}} = -\frac{a-x}{v} = \frac{LO}{Om}$.

Example VII. *Fig.* 69.

174. *To find the radius of Curvature in the Quadratrix.*

Call GO, x, OP, y, GA, r, then the ordinate OP is equal to the arc AF and by art. 125. $\frac{\dot{y}}{r} \times \overline{x^2 + y^2} = x\dot{y} - y\dot{x}$ or $\frac{z}{r} \times \overline{x^2 + y^2} = xz - y$ put $x^2 + y^2 = v^2$ then $\frac{zv^2}{r} = xz - y$ or $v^2 = rx -$ $\frac{ry}{z}$ and taking the fluxions $2v\dot{v} = r\dot{x} - \frac{r\dot{y}}{z} +$ $\frac{ry\dot{z}}{zz}$. But $2v\dot{v} = 2x\dot{x} + 2y\dot{y} = 2x\dot{x} + 2yz\dot{x}$ because $\dot{y} = z\dot{x}$ therefore $2x\dot{x} + 2y\dot{x}z =$ $r\dot{x} - r\dot{x} + \frac{ry\dot{z}}{zz} = \frac{ry\dot{z}}{zz}$ and $\frac{\dot{z}}{\dot{x}} = \frac{2z^2x + 2yz^3}{yr} =$

$2y^3 +$

$\dfrac{2y^3+\dfrac{2xy^2}{z}}{\dfrac{y^3r}{z^3}}$. But $\dfrac{y}{z}=\dfrac{y\dot{x}}{\dot{y}}=-\text{OT}$ therefore

$\dfrac{\dot{z}}{\dot{x}}=\dfrac{2y^3-2xy\times\text{OT}}{r\times\text{OT}^3}$ and the radius of curvature

$\text{CP}=\dfrac{\text{PT}^3}{\text{OT}^3}\times\dfrac{\dot{x}}{\dot{z}}=\dfrac{\text{PT}^3\times r}{2y\times\overline{y^2-x\times\text{OT}}}=$

$\dfrac{\text{PT}^3\times\text{GA}}{2\text{OP}\times\overline{\text{OP}^2-\text{GO}\times\text{OT}}}$. But $\text{OP}^2=\text{OT}\times\text{OD}$

and $\text{OP}^2-\text{GO}\times\text{OT}=\text{OT}\times\text{GD}$, also $\text{PT}^2=$

$\text{DT}\times\text{OT}$ therefore $\text{CP}=\dfrac{\text{PT}\times\text{DT}\times\text{OT}\times\text{GA}}{2\text{OP}\times\text{OT}\times\text{GD}}=$

$\dfrac{\text{PT}\times\text{DT}\times\text{GA}}{2\text{OP}\times\text{GD}}$ and $\text{PE}=\dfrac{\text{OP}\times\text{CP}}{\text{PD}}=$

$\dfrac{\text{PT}\times\text{DT}\times\text{GA}}{2\text{PD}\times\text{GD}}$. And $\text{CE}=\dfrac{\text{OP}\times\text{CP}}{\text{PT}}=$

$\dfrac{\text{DT}\times\text{GA}}{2\text{GD}}$ therefore 2GD : GA : DT : CE.

Example VIII. *Fig* 51.

175. *To find the radius of Curvature in the curve of swiftest descent.*

Call BO, x, OP, y and the fluxion of the arc AP, $\dot{t}$, then by art. 143. $\dfrac{\dot{y}^2}{\dot{t}^2x^n}$ is a given quantity.

Let it $= \frac{1}{a^n}$ then $a^n\dot{y}^2 = x^n\dot{t}^2 = x^n \times \overline{\dot{x}^2 + \dot{y}^2}$ or $a^n z^2 = x^n + x^n z^2$ and $\frac{a^n}{x^n} = \frac{1}{z^2} + 1$ and taking the fluxions $\frac{na^n\dot{x}}{x^{n+1}} = \frac{2\dot{z}}{z^3}$ and $\frac{\dot{x}}{\dot{z}} = \frac{2x^{n+1}}{na^n z^3}$. Now since $\frac{a^n}{x^n} = \frac{z^2+1}{z^2}$ we have $\frac{\dot{t}^2}{\dot{x}^2} = 1 + z^2 = \frac{a^n z^2}{x^n}$ and $\frac{\dot{t}^3}{\dot{x}^3} = \frac{z^3 a^{\frac{3n}{2}}}{x^{\frac{3n}{2}}}$ and the radius of curvature $cP =$ $\frac{\dot{t}^3}{\dot{x}^3} \times \frac{\dot{x}}{\dot{z}} = \frac{2a^{\frac{n}{2}}}{x^{\frac{n}{2}-1}}$. And $cE = \frac{\dot{t}^2}{\dot{x}^2} \times \frac{\dot{x}z}{\dot{z}} =$ $\frac{2x}{n}$, whence $cQ = cE - QE = \frac{2x}{n} - x =$ $\frac{2-n}{n} \times x$ and $cR : cP :: cQ : cE :: 2-n : 2$.

Lemma VI.

176. *Let* SP*p* *(fig.* 52.) *be a spiral,* P, *p*, *two points in it infinitely near each other* ; PT, *pt two tangents in the points* P, *p* ; ST, S*t perpendicular to* SP, S*p* ; *from the point* T *let fall the perpendicular* TK *on* P*t*, *to find the ultimate ratio of* TK *to* P*p*.

From the point S let fall the perpendicular SY on the tangent *pt* meeting the tangent PT in *y*. Call SP, *y* SF, *r*, the fluxion of the arc BF, $\dot{x}$, the fluxion of the curve, $\dot{t}$, and the perpendicular SY,

SY, z. Then since yY and TK are parallel, TK : yY :: PT : PY :: PT^2 : SP^2 whence TK $= \frac{yY \times PT^2}{SP^2}$ and $\frac{Pp}{TK} = \frac{Pp}{yY} \times \frac{SP^2}{PT^2} = \frac{\dot{t}}{\dot{z}} \times \frac{SP^2}{PT^2}$.

Q. E. I.

Prop. XXXIII.

177. *To find the radius of curvature in a spiral.*

Let C (fig. 52.) be the center of curvature, then since TK and Cp are both perpendicular to the same right line pt they will be parallel, and the angle CpP will be equal to the angle pTK and as the right angles CPp, pKT are also equal, the triangles CpP, pTK will be similiar and Cp : Pp :: pT : KT and CP $= \frac{Pp \times pT}{KT} =$ (by Lemma VI.) $\frac{\dot{t}}{\dot{z}} \times \frac{SP^2}{PT}$ but TP $= \frac{Pp \times SP}{pn} = \frac{y\dot{t}}{\dot{y}}$ therefore the radius of curvature CP $= \frac{\dot{t}}{\dot{z}} \times \frac{y^2\dot{y}}{y\dot{t}} = \frac{y\dot{y}}{\dot{z}}$.

Q. E. I.

Corol. Since the triangles SPY, PCE are similiar we have SP : SY :: CP : PE and PE $= \frac{1}{2}$ the chord passing through S $= \frac{CP \times SY}{SP} = \frac{z\dot{y}}{\dot{z}}$.

178. If the arc BF be as y^m or $x : r :: y^m : a^m$ then by art. 122. $z^2 = \frac{m^2y^{2m+2}}{a^{2m} + m^2y^{2m}}$ or $a^{2m} + m^2y^{2m}$

=

$=\frac{m^2y^{2m+2}}{z^2}$ and taking the fluxions $m\dot{y}y^{2m-1}=$ $\frac{\overline{m+1}\times\dot{y}y^{2m+1}}{z^2}-\frac{y^{2m+2}\dot{z}}{z^3}$ and multiplying by $\frac{z^3}{y^{2m-1}}$ we have $m\dot{y}z^3=\overline{m+1}\times\dot{y}zy^2-y^3\dot{z}$ and $\frac{\dot{y}}{\dot{z}}=$ $\frac{y^3}{z\times\overline{m+1}\times y^2-mz^2}$ and the radius of curvature $CP=\frac{y\dot{y}}{\dot{z}}=\frac{y^4}{z\times\overline{\overline{m+1}\times y^2-mz^2}}$. And $PE=$ $\frac{z\dot{y}}{\dot{z}}=\frac{y^3}{\overline{m+1}\times y^2-mz^2}$.

In the spiral of Archimedes $m=1$, $CP=$ $\frac{y^4}{z\times 2y^2-z^2}$ and $PE=\frac{y^3}{2y^2-z^2}$.

In the reciprocal spiral $m=-1$. $CP=\frac{y^4}{z^3}$ and $PE=\frac{y^3}{z^2}$.

In the Lituus, $m=-2$. $CP=\frac{y^4}{z\times 2z^2-y^2}$ and $PE=\frac{y^3}{2z^2-y^2}$.

179. Let SP (fig. 52.) be the equiangular spiral, then by art. 124. $a\dot{y}=y\dot{x}$ and z is to y in a given ratio as suppose $c:r$ then $z=\frac{cy}{r}$, $\dot{z}=\frac{c\dot{y}}{r}$ and the radius of curvature $CP=\frac{y\dot{y}}{\dot{z}}=\frac{ry}{c}$ and $PE=$ $\frac{z\dot{y}}{\dot{z}}=\frac{rz}{c}=y=PS$.

Since $y : z :: \dot{t} : \frac{y\dot{x}}{r}$ we have $r : c :: \dot{t} : \frac{y\dot{x}}{r}$ and $\dot{t} = \frac{y\dot{x}}{c} = \frac{a\dot{y}}{c}$ and as the fluxion of the radius of curvature is $\frac{r\dot{y}}{c}$ the variation of curvature $=$ $\frac{\dot{CP}}{\dot{t}} = \frac{r\dot{y}}{c\dot{t}} = \frac{r}{a} =$ a given quantity.

180. Let AP (fig. 20.) be an Epicycloid, let SY be perpendicular to the tangent VP and PO perpendicular to SV. Call SB, b, BD, c, SP, y, SY, z, and BO, s, then (by 12. 2 Elem.) $SP^2 =$ $SB^2 + BP^2 + 2SB \times BO = SB^2 + BO \times \overline{2BD + 2SB}$ or $y^2 = b^2 + 2s \times \overline{b + c}$ and taking the fluxions $y\dot{y} = \dot{s} \times \overline{b + c}$. Also because BP and SY are parallel, $VB^2 : VS^2 :: BP^2 : SY^2$ and $SY^2 =$ $\frac{VS^2 \times BP^2}{VB^2} = \frac{VS^2 \times VB \times BO}{VB^2} = \frac{VS^2 \times BO}{VB}$ or $z^2 = \frac{\overline{b + 2c}|^2 \times s}{2c}$ and taking the fluxions $2z\dot{z} =$ $\frac{\overline{b + 2c}|^2 \times \dot{s}}{2c} = \frac{z^2\dot{s}}{s}$ and $\dot{z} = \frac{z\dot{s}}{2s}$ therefore the radius of curvature $CP = \frac{y\dot{y}}{\dot{z}} = \frac{\dot{s} \times \overline{b + c} \times 2s}{z\dot{s}} =$ $\frac{\overline{b + c} \times 2s}{z} = \frac{2SD \times BO}{SY}$. But BO : BP :: BP : BV :: SY : VS therefore $\frac{BO}{SY} = \frac{BP}{VS}$ and $CP =$ $\frac{2SD \times BP}{VS} = \frac{BP \times \overline{SB + SV}}{VS}$. And VS : VS + SB :: BP : CP.

Let

Let v (fig. 53.) be the vertex, then $BP = Bv$ and $vS : vS + SB :: Bv : Cv$ and disjointly $vS : SB :: Bv : CB$ or $vS : Bv :: SB : CB$ and again disjointly $Sv : SB :: SB : SC$.

PROP. XXXIV.

181. *Let* ARB *(fig. 54.) be an ellipse and let the curve* APE *be formed by taking any point* R *in the ellipse and making* PS=SR *in such manner that the angle* ASR *may be to the angle* ASP *in a given ratio as* m *to* n, *to find the radius of curvature**.

Let SN be the conjugate diameter to SR and let fall the perpendicular RF from R on SN.

Call AS, a, SB, b, $SR = SP, y$, SN, v, and the perpendicular SY let fall from S on the tangent PY, z. Let the semidiameter Sr be infinitely near to SR, let $Sp = Sr$ and on SP, SR let fall the perpendiculars pn, rq.

Since the triangles Rrq, SRF are similiar, we have $Rq^2 : rq^2 :: SF^2 : RF^2 :: SR^2 - RF^2 : RF^2 :: SR^2 - \frac{AS^2 \times SB^2}{SN^2} : \frac{AS^2 \times SB^2}{SN^2} :: y^2v^2 - a^2b^2 : a^2b^2$ and $Rq^2 = Pn^2 = \frac{y^2v^2 - a^2b^2}{a^2b^2} \times rq^2$. Also since the angle ASR is to the angle ASP as m to n, and ASr is to ASp in the same ratio, therefore the angle $RSr : PSp :: rq : pn :: m : n$ and $rq^2 = \frac{m^2}{n^2} \times pn^2$ therefore $Pn^2 = \frac{m^2y^2v^2 - m^2a^2b^2}{n^2a^2b^2} \times pn^2$,

* See Newton. Princip. Lib. III. Prop. XXVIII.

and $Pp^2 = Pn^2 + pn^2 = pn^2 \times \frac{m^2y^2v^2 + n^2a^2b^2 - m^2a^2b^2}{n^2a^2b^2}$; put $nn - mm = p$ then $\frac{Pp^2}{pn^2} = \frac{m^2y^2v^2 + pa^2b^2}{n^2a^2b^2}$. But as the triangles SYP, pnP are similiar, SY : SP :: pn : Pp : and SY² or $z^2 = \frac{SP^2 \times pn^2}{Pp^2} = \frac{n^2a^2b^2y^2}{m^2y^2v^2 + pa^2b^2}$, therefore $m^2y^2v^2z^2 + pa^2b^2z^2 = n^2a^2b^2y^2$ or $\frac{n^2a^2b^2}{z^2} = m^2v^2 + \frac{pa^2b^2}{y^2}$ and taking the fluxions $\frac{n^2a^2b^2\dot{z}}{z^3} = \frac{pa^2b^2\dot{y}}{y^3} - m^2v\dot{v}$. But by conics $y^2 + v^2 = a^2 + b^2$ therefore $v\dot{v} = - y\dot{y}$ and $\frac{n^2a^2b^2\dot{z}}{z^3} = \frac{pa^2b^2\dot{y} + m^2y^4\dot{y}}{y^3}$ and $\frac{\dot{y}}{\dot{z}} = \frac{y^3}{z^3} \times \frac{n^2a^2b^2}{pa^2b^2 + m^2y^4}$ and CP the radius of curvature $= \frac{y\dot{y}}{\dot{z}} = \frac{n^2a^2b^2y^4}{z^3 \times pa^2b^2 + m^2y^4}$.

Corol. 1. When $y = a$, $v = b$ and $z^2 = \frac{n^2a^4b^2}{m^2a^2b^2 + pa^2b^2} = a^2$ and the radius of curvature in A $= \frac{n^2a^6b^2}{a^3 \times pa^2b^2 + m^2a^4} = \frac{ab^2n^2}{pb^2 + m^2a^2}$.

When $y = b$, $v = a$ and $z = b$ and the radius of curvature in E where SE = SB is $\frac{n^2a^2b^6}{b^3 \times pa^2b^2 + m^2b^4} = \frac{ba^2n^2}{pa^2 + m^2b^2}$. Therefore the

radius

radius of curvature in A is to the radius of curvature in E as $\frac{ab^2n^2}{pb^2+m^2a^2}$ to $\frac{ba^2n^2}{pa^2+m^2b^2}$ or as $pa^2b+m^2b^3 : pb^2a+m^2a^3 :: b^3+\frac{p}{m^2}\times a^2b : a^3+\frac{p}{m^2}\times ab^2$ and the curvature in A is to the curvature in E as $a^3+\frac{p}{m^2}\times ab^2$ to $b^3+\frac{p}{m^2}\times a^2b$.

Corol. 2. The curvature in A is to the curvature of a circle whose radius is AS as $\frac{pb^2+m^2a^2}{ab^2n^2} : \frac{1}{a} :: pb^2+m^2a^2 : n^2b^2$ and disjointly the difference between the curvatures of the curve in A, and the circle is to the curvature of the circle as $m^2a^2-m^2b^2$ to n^2b^2.

Since the radius of curvature of the ellipse in A is $\frac{SB^2}{SA}$ or $\frac{b^2}{a}$ the curvature of the ellipse in A will be to the curvature of a circle whose radius is SA as $a^2 : b^2$ and the curvature of this circle is to the difference between the curvatures of the ellipse in A and this circle as $b^2 : a^2-b^2$.

Therefore *ex æquo* the difference between the curvature of the curve in A and this circle is to the difference between the curvatures of the ellipse in A and the same circle as m^2 to n^2.

In the same manner the difference between the curvatures of the curve in E, and of a circle whose radius is SE or SB is to the difference between the curvatures of the ellipse in B and the same circle, as m^2 to n^2.

Prop. XXXV.

182. *To find a curve in which the radius of curvature is inversely as the power m of the ordinate.*

Let the abscissa be x, ordinate y, and the fluxion of the curve $\dot{t}$, then the radius of curvature will be (by Prop. XXX.) $= \frac{\dot{y}\dot{t}}{\ddot{t}\dot{x} - \dot{t}\ddot{x}}$ let $\dot{t}$ be invariable then the radius $= \frac{\dot{y}\dot{t}}{\ddot{x}}$ which is to be as $\frac{1}{y^m}$ or if a is a given quantity let it be equal $\frac{a^{m+1}}{y^m}$ then $y^m \dot{y}\dot{t} = a^{m+1}\ddot{x}$ and taking the fluents $\frac{y^{m+1}\dot{t}}{m+1} = a^{m+1}\dot{x} + C\dot{t}$ where $C\dot{t}$ is a given quantity.

If $C = 0$, the equation is $\frac{y^{m+1}\dot{t}}{m+1} = a^{m+1}\dot{x}$ or $y^{m+1}\dot{t} = \overline{m+1} \times a^{m+1}\dot{x}$ and squaring both sides of the equation $\overline{m+1}|^2 \times a^{2m+2}\dot{x}^2 = y^{2m+2}\dot{t}^2 = y^{2m+2}\dot{x}^2 + y^{2m+2}\dot{y}^2$ and $y^{2m+2}\dot{y}^2 = \dot{x}^2 \times \overline{\overline{m+1}|^2 \times a^{2m+2} - y^{2m+2}}$.

If $m=2$ the equation is $y^6\dot{y}^2 = \dot{x}^2 \times \overline{9a^6 - y^6}$.

If $m=1$ the equation is $y^4\dot{y}^2 = \dot{x}^2 \times \overline{4a^4 - y^4}$.

Which curve is called the elastic curve.

If $m = 0$, that is, if the radius is constant and equal to a, the equation will be $y^2\dot{y}^2 = \dot{x}^2 \times \overline{a^2 - y^2}$ or $\dot{x} = \frac{y\dot{y}}{\sqrt{a^2 - y^2}}$ and taking the fluents $x = \sqrt{a^2 - y^2}$ which shews the curve to be a circle whose radius is a.

Sect. VII.

Of the Points of contrary Flexure and Reflection.

183. When one part of a curve is concave, and the other part convex towards the base, and the curve is continued beyond the ordinate that separates these parts, then this ordinate is said to pass through a point of contrary flexure.

Prop. XXXVI.

184. *To find when an ordinate passes through a point of contrary flexure.*

Let P (fig. 45.) be a point of contrary flexure, OP an ordinate drawn through it, A the beginning of the abscissa, and PT a tangent in the point P.

Call AO, x, OP, y, then the distance of the point T from A will be a maximum or a minimum, for if the part of the curve next to T, be concave to the base, AT will be increasing while the tangent passes from π to P and will be diminishing while it passes from P to p, and therefore AT will be a maximum.

If the part of the curve next to T be convex to the base, then AT will be diminishing while the tangent passes from π to P and will be increasing while it passes from P to p and consequently AT will be a minimum.

Therefore $AT = OT - AO = \frac{y\dot{x}}{\dot{y}} - x$ will be a maximum or minimum, and taking the fluxions Suppofing $\dot{x}$ invariable $\dot{x} - \frac{y\dot{x}\ddot{y}}{\dot{y}^2} - \dot{x}$ or $\frac{-y\dot{x}\ddot{y}}{\dot{y}^2}$ will be either o or infinite (Prop. XXVI.) and therefore $\ddot{y}$ will be either o or infinite.

Take $Oo = O\omega = \dot{x}$ on each fide of OP, and let the ordinates op, $\omega\pi$, meet the curve in p, π, and the tangent PT in t, τ, and fince $\dot{x} : \dot{y} :: TO : OP :: T\omega : \omega\tau :: To : ot$ we have $\omega\tau = \frac{T\omega \times \dot{y}}{\dot{x}} =$

$$\frac{\dot{y} \times \overline{\frac{y\dot{x}}{\dot{y}} - \dot{x}}}{\dot{x}} = y - \dot{y} \text{ and } ot = \frac{To \times \dot{y}}{\dot{x}} =$$

$$\frac{\dot{y} \times \overline{\frac{y\dot{x}}{\dot{y}} + \dot{x}}}{\dot{x}} = y + \dot{y}.$$ And by Prop. XXV. $op = y + \dot{y} + \frac{1}{2}\ddot{y} + \frac{1}{6}\dddot{y} + \frac{1}{24}\ddddot{y}$, &c. $= ot + \frac{1}{2}\ddot{y} + \frac{1}{6}\dddot{y} + \frac{1}{24}\ddddot{y}$, &c. and $\omega\pi = y - \dot{y} + \frac{1}{2}\ddot{y} - \frac{1}{6}\dddot{y} + \frac{1}{24}\ddddot{y}$, &c. $= \omega\tau + \frac{1}{2}\ddot{y} - \frac{1}{6}\dddot{y} + \frac{1}{24}\ddddot{y}$, &c.

If $\ddot{y} = o$ and $\dddot{y}$ does not vanifh, then $op = ot + \frac{1}{6}\dddot{y}$ &c. and $\omega\pi = \omega\tau - \frac{1}{6}\dddot{y}$, &c. Therefore the curve Pp will lie above the tangent PT becaufe op is greater than ot, and at the fame time the part $P\pi$ will fall below the tangent PT becaufe $\omega\tau$ is lefs than $\omega\tau$; and therefore P muft be a point of contrary flexure.

If $\ddot{y}$ and $\dddot{y}$ vanifh and $\ddddot{y}$ does not vanifh then $op = ot + \frac{1}{24}\ddddot{y}$ and $\omega\pi = \omega\tau + \frac{1}{24}\ddddot{y}$ and therefore both

both the parts of the curve Pp, $P\pi$, will lie above the tangent TP because op is greater than ot, and $\omega\pi$ greater than $\omega\tau$, and therefore the point P will not be a point of contrary flexure.

In general if $\ddot{y}$ and any number of fluxions of the next successive orders vanish, the point P will be a point of contrary flexure or not, according as the number of fluxions that vanish is odd or even.

185. If $\ddot{y}$ is infinite, and if the signs of $\ddot{y}$, before and after the ordinate has passed the point P be different, then the line AT will be a maximum or minimum, and the point P will be a point of contrary flexure, if the curve be continued both sides of the ordinate.

If the signs of $\ddot{y}$ be the same, then P is not a point of contrary flexure.

Corol. 1. When an ordinate passes through a point of contrary flexure, its first fluxion is a maximum or minimum, because an odd number of its fluxions vanish. (Prop. XXVI.)

Corol. 2 The radius of curvature in a point of contrary flexure is either infinite or nothing. For it appears by Prop. XXX. that when $\dot{x}$ is invariable the radius of curvature is $\frac{\dot{t}^3}{\dot{x}\ddot{y}}$ ($\dot{t}$ being the fluxion of the curve) but in a point of contrary flexure, $\ddot{y}$ is either o or infinite therefore the radius of curvature must be either infinite or o, and the curvature o or infinite.

Corol. 3. Therefore in a point of contrary flexure, the evoluta either meets the curve in that point,

point, or touches the perpendicular to the curve at an infinite distance, that is the perpendicular either touches the evoluta in that point or is an asymptote to it.

Example I. *Fig.* 8.

186. *To find the point of contrary flexure in the* Conchoid *of* Nicomedes.

Retaining the same symbols as in art. 168. we have $\frac{\dot{z}}{\dot{x}} = \frac{a^2 \times \overline{q - 3bv^2}}{x^3 v^3}$. But as $z = \frac{\dot{y}}{\dot{x}}$ and $\dot{x}$ is invariable we have $\dot{z} = \frac{\ddot{y}}{\dot{x}}$ therefore $\frac{\ddot{y}}{\dot{x}^2} = \frac{a^2}{x^3 v^3} \times \overline{q - 3bv^2}$.

Put $\ddot{y} = 0$ then $q - 3b\,v^2 = 0$ or substituting the values of q and v; $a^2 b + x^3 - 3a^2 b + 3bx^2 = 0$ or $x^3 + 3bx^2 - 2a^2 b = 0$ by the resolution of which equation we may find x or GO and consequently the point P.

Example II. *Fig.* 47.

187. *To find the point of contrary flexure in a parabola whose equation is* $y = ax^m$.

By taking the fluxions we have $\frac{\ddot{y}}{\dot{x}^2} = m \cdot \overline{m - 1} \cdot ax^{m-2}$ which becomes $= 0$ when $x = 0$, if m is greater than 2. By taking the fluxions again we have

have $\frac{\dddot{y}}{\dot{x}^3} = m \cdot m-1 \cdot m-2 \cdot ax^{m-3}$. $\frac{\ddddot{y}}{\dot{x}^4} =$

$m \cdot m-1 \cdot m-2 \cdot m-3 \cdot ax^{m-4}$.

If $m=3$, then $\dddot{y}$ does not vanish, and the point P is a point of contrary flexure.

If $m=4$, then $\dddot{y}=0$, and $\ddddot{y}$ does not vanish, and P is not a point of contrary flexure.

And in general if m is greater than 2, and is an odd number, the point P where the curve cuts the base is a point of contrary flexure, otherwise not.

Example III. *Fig.* 17.

188. *To find the point of contrary flexure in a cycloid.*

Retaining the same symbols as in art. 172. we have $\frac{\ddot{y}}{\dot{x}^2} = \frac{\dot{z}}{\dot{x}} = \frac{a^2+bx}{v^3}$. And if $\ddot{y}=0$, $a^2+bx=0$, and $x=\frac{-a^2}{b}$ and $v=\frac{a}{b}\sqrt{b^2-a^2}$. Now since $\frac{\ddot{y}}{\dot{x}^2} = \frac{a^2+bx}{v^3}$ we have $\frac{\ddot{y}v^3}{\dot{x}^2} = a^2+bx$ and taking the fluxions $\frac{\dddot{y}v^3+3\ddot{y}\dot{v}v^2}{\dot{x}^2} = b\dot{x}$ or since $\ddot{y}=0$, $\frac{\dddot{y}v^3}{\dot{x}^2} = b\dot{x}$ and $\dddot{y} = \frac{b\dot{x}^3}{v^3} = \frac{b^4\dot{x}^3}{a \times \overline{b^2-a^2}|^{\frac{3}{2}}}$.

If a is greater than b, v will be impossible, and and if $a=b$, $v=0$, and $\dddot{y}$ is infinite, and therefore in both these cases there can be no point of contrary flexure.

But

But if b is greater than a, then $\dot{y}$ will not vanish, and the curve will have a point of contrary flexure, when x is a third proportional to b and a, therefore take CA : CH :: CH : Co and raise the ordinate op, which will meet the curve in the point of contrary flexure.

Corol. Let op meet the circle HMK in μ then Aμ will be a tangent to the circle HMK for CA : Cμ :: Cμ : Co.

Prop. XXXVII.

189. *To find the point of contrary flexure in a spiral.*

Let P (fig. 23.) be a point of contrary flexure, PT a tangent in the point P, SY a perpendicular let fall on it from the point S. Then it is plain that SY is a maximum or minimum, now by art. 121. SY $= \frac{y^2 \dot{x}}{r\dot{t}}$ where $y =$ SP, $x =$ BF, $r =$ BS, and $\dot{t} =$ the fluxion of the arc SP, put SY $= z$ then (by Prop. XXVI.) $\dot{z}$ is either o or infinite.

If $\dot{z}$ and any even number of its fluxions of subsequent orders vanish, then z is a maximum or minimum, and the point P is a point of contrary flexure, if the curve is continued both sides of the line SP.

If $\dot{z}$ is infinite, we may determine in the same manner as in art. 134. whether z is a maximum or minimum, and P a point of contrary flexure.

190. Let $x : r :: y^m : a^m$ then by art. 122. $z = \frac{m^2 y^{2m+2}}{a^{2m} + m^2 y^{2m}}$ and $\frac{\dot{z}}{\dot{y}}$ (art. 178.) $=$

z ×

4

$\frac{z \times \overline{\overline{m+1} \, . \, y^2 - mz^2}}{y^3}$ which to be $= 0$ or infinite and therefore $\overline{m+1} \, . \, y^2 - mz^2 = 0$ (since z and y must not be infinite) Therefore $y^2 = \frac{mz^2}{m+1} = \frac{m^2 y^{2m+2}}{\overline{m+1} \times \overline{a^{2m} + m y^{2m}}}$ and $\overline{m+1} \times a^{2m} + \overline{m^3 + m^2} \times y^{2m} = m^3 y^{2m}$ whence $y^{2m} = -\frac{m+1}{m^2} \times a^{2m}$ therefore y will be impossible except $m+1$ is negative or m must be less than -1.

Since $\frac{\dot{z}}{\dot{y}} = \frac{z \times \overline{\overline{m+1} \, . \, y^2 - mz^2}}{y^3}$ we have $\frac{\dot{z}}{\dot{y}z} = \frac{\overline{m+1} \, . \, y^2 - mz^2}{y^3} = \frac{m+1}{y} - \frac{mz^2}{y^3}$ and taking the fluxions, supposing $\dot{y}$ invariable, and $\dot{z} = 0$, then $\frac{\ddot{z}}{\dot{y}z} = -\frac{m+1}{y^2} \times \dot{y} + \frac{3mz^2\dot{y}}{y^4}$ but $\frac{mz^2}{y^2} = m+1$, therefore $\frac{\ddot{z}}{z\dot{y}^2} = \frac{-m-1+3m+3}{y^2} = \frac{2m+2}{y^2}$, therefore $\ddot{z}$ does not vanish and z is a maximum, since $m+1$ is negative, and the point P where $y = a \sqrt[2m]{-\frac{m+1}{m^2}}$ is a point of contrary flexure.

In the Lituus, $m = -2$ and $y = a \sqrt[-4]{\tfrac{1}{4}} = a\sqrt{2}$.

Of

Of the Points of Reflection or Cusps.

191. If a curve inſtead of being continued beyond the ordinate is reflected from it, both parts of the curve having the ſame tangent, different from the ordinate, then that ordinate is ſaid to paſs through a point of Reflection or Cuſp. (fig. 32, 33, 34.)

It is called a Cuſp of the firſt kind, when the two parts of the curve have their convexities turned towards each other. (fig. 34.) And it is called a Cuſp of the ſecond kind, when the convexity of one is towards the concavity of the other (fig. 33.)

192. Whenever the curve is not continued beyond the ordinate and does not touch it, it muſt form a Cuſp, and if the tangent is not parallel to baſe, the ordinate will be a maximum or minimum of the ſecond kind, and the point may be determined by Prop. XXVII.

If the tangent is parallel to the baſe, it may be found by putting $\dot{y}=0$.

To determine of what kind the Cuſp is, if it is of the firſt kind, the fluxions of the two values of AT will have different ſigns, that is, the ſecond fluxions of the two values of y will have different ſigns.

If it is a Cuſp of the ſecond kind, they will have the ſame ſigns.

193. Let

193. Let dPD (fig. 55.) be a perpendicular to the curve AP in the point P, and let $GHgb$ be the evoluta to the curve. Then if dPD is an asymptote to the curve $GHgb$, and if the two parts of the curve $GHgb$ be on the same side of the right line Dd the curve will have a Cusp in P, which will be of the first kind, if the two parts of the evoluta GH, gb, be on different sides of the point P, and it it will be of the second kind when GH, gb, are on the same side of P. And in both cases the radius of curvature will be infinite, and therefore if $\dot{x}$ is invariable, $\frac{\dot{t}^3}{\dot{x}\ddot{y}}$ will be infinite, and $\ddot{y} = 0$.

194. If the evoluta GH (fig. 56.) pass through P, and have a continued curvature at P, then the curve AP will have a Cusp of the first kind in P.

But if GH has a Cusp of the second kind at P, the curve AP will also have a Cusp of the second kind at P. And the radius of curvature $\frac{\dot{t}^3}{\dot{x}\ddot{y}} = 0$ and $\ddot{y}$ will be infinite.

Example I. *Fig.* 57.

To find where the Cissoid *of* Diocles *has a Cusp.*

195. Supposing the same things as in art. 167. the cissoid is not continued beyond A and the radius of curvature in $A = 0$, therefore the curve has a Cusp at the point A, which is of the first kind because the two values of y are $\pm\sqrt{\frac{x^3}{a-x}}$ and consequently the two values of $\ddot{y}$, have different signs.

Example

Example II. *Fig.* 57.

196. *To find when a Parabola whose equation is $y = ax^m$ has a Cusp.*

When m is half an odd number, the curve cannot be continued beyond A, for otherwise $y = a \times \overline{-x}|^m$ would be impossible, and by art. 170. the radius of curvature in $A = \frac{my^2}{x} = ma^2\, x^{2m-1}$ which when m is greater than $\frac{1}{2}$ becomes $= 0$ when $x = 0$, and since $y = \pm \sqrt{a^2 x^{2m}}$ the two values of $\ddot{y}$ will have different signs, therefore the curve has a Cusp of the first kind at A.

Example III. *Fig.* 92.

197. *To find the Cusp in the curve whose equation is $y = \sqrt{x} \pm \sqrt[4]{x^3}$.*

This curve cannot be continued beyond A, for otherwise y would be impossible, by taking the fluxions we have $\dot{y} = \frac{1}{2} \times \dot{x} x^{-\frac{1}{2}} \pm \frac{3}{4} \times \dot{x} x^{-\frac{1}{4}}$ and $\ddot{y} = \frac{-1}{4} \times \dot{x}^2 x^{-\frac{3}{2}} \mp \frac{3}{16}\, \dot{x}^2\, x^{-\frac{5}{4}}$. And when $x = 0$, $\ddot{y}$ is infinite, but when x is not $= 0$ but very small, $\ddot{y}$ is nearly $= \frac{-\dot{x}^2}{4\sqrt{x^3}}$ which must have but one sign, because if $\sqrt{x}$ was negative $\sqrt[4]{x^3} = \sqrt{x} \times \sqrt[4]{x}$ would be impossible, therefore both the values of $\ddot{y}$ have the same sign, and therefore the curve will have a Cusp of the second kind in A.

SECT.

Sect. VIII.

Of the Quadrature of Curves.

Prop. XXXVIII.

198. If $\dot{A} = \dot{x}\, x^{\theta-1} \times \overline{e+fx^{n}}|^{\lambda-1}$ and $\dot{B} = \dot{A}x^{n} = \dot{x}x^{\theta+n-1} \times \overline{e+fx^{n}}|^{\lambda-1}$ *I say that* $\theta e\mathrm{A} + \overline{\theta+\lambda n} \times f\mathrm{B} = x^{\theta} \times \overline{e+fx^{n}}|^{\lambda}$ *.

Put $e+fx^{n} = z$, $x^{\theta-1}\,\dot{x} = \dot{v}$ and $z^{\lambda} = y$ then $v = \frac{1}{\theta}x^{\theta}$ and $\dot{y} = \lambda\dot{z}z^{\lambda-1} = \lambda nf\dot{x}x^{n-1}\, z^{\lambda-1}$ (art. 20.) therefore $\dot{v}y = \dot{A}z = e\dot{A} + fx^{n}\dot{A} = e\dot{A} + f\dot{B}$, alſo $\dot{y}\,v = \frac{\lambda nf}{\theta} \times \dot{x}x^{\theta+n-1}\, z^{\lambda-1} = \frac{\lambda nf\dot{B}}{\theta}$, therefore $\dot{v}y + \dot{y}v = e\dot{A} + \frac{\theta+\lambda n}{\theta} \times f\dot{B}$ and taking the fluents $\overline{vy+yv}| = e\mathrm{A} + \frac{\theta+\lambda n}{\theta} \times f\mathrm{B}$ but $yv = \frac{1}{\theta}x^{\theta}z^{\lambda}$ therefore $e\mathrm{A} + \frac{\theta+\lambda n}{\theta} \times f\mathrm{B} = \frac{1}{\theta} \times x^{\theta}z^{\lambda}$ and $\theta e\mathrm{A} + \overline{\theta+\lambda n} \times f\mathrm{B} = x^{\theta}z^{\lambda}$: Q. E. D.

Corol. 1. Put $\dot{x}x^{\theta-1} \times \overline{e+fx^{n}}|^{\lambda} = \dot{\mathrm{F}}$ then $\dot{\mathrm{F}} = \dot{A}z = e\dot{A} + f\dot{B}$ and $\mathrm{F} = e\mathrm{A} + f\mathrm{B}$.

Corol. 2. If $\dot{A} = \dot{x}x^{\theta-1} \times \overline{e+fx^{n}}|^{\lambda-1}$, $\dot{B} = \dot{A}x^{n}$, $\dot{C} = \dot{B}x^{n} = \dot{A}x^{2n}$, $\dot{D} = \dot{C}x^{n} = \dot{A}x^{3n}$, then

* Cotes Harmon. Menſur. p. 66.

$\theta e\mathrm{A} + \overline{\theta + \lambda\eta}\,.\, f\mathrm{B} = x^{\theta}z^{\lambda}$; and as $\dot{\mathrm{B}} = \dot{x}x^{\theta+\eta-1}z^{\lambda-1}$, for A, B, θ, in the foregoing theorem write B, C, $\theta + \eta$ and then $\overline{\theta+\eta}\,.\,e\mathrm{B} + \overline{\theta + \lambda\eta + \eta}\,.\,f\mathrm{C} = x^{\theta+\eta}z^{\lambda}$, and in the same manner $\overline{\theta+2\eta}\,.\,e\mathrm{C} + \overline{\theta + \lambda\eta + 2\eta}\,.\,f\mathrm{D} = x^{\theta+2\eta}z^{\lambda}$.

Corol. 3. If we put $\theta+\lambda\eta = s$, then $\theta e\mathrm{A} + sf\mathrm{B} = x^{\theta}z^{\lambda}$; $\overline{\theta+\eta}\,.\,e\mathrm{B} + \overline{s+\eta}\,.\,f\mathrm{C} = x^{\theta+\eta}z^{\lambda}$; and $\overline{\theta+2\eta}\,.\,e\mathrm{C} + \overline{s+2\eta}\,.\,f\mathrm{D} = x^{\theta+2\eta}z^{\lambda}$. And if $x^{\eta} = \frac{-e}{f}$ then $z = 0$ and the fluents B, C, D generated while x flows from 0 to $\frac{-e}{f}\big|^{\frac{1}{\eta}}$ will be

$$\mathrm{B} = -\frac{\theta e}{sf}\mathrm{A},\ \mathrm{C} = -\frac{\theta+\eta}{s+\eta}\times\frac{e}{f}\mathrm{B} = +\frac{\theta\,.\,\theta+\eta}{s\,.\,s+\eta}\times\frac{e^2}{f^2}\mathrm{A},\ \mathrm{D} = -\frac{\theta+2\eta}{s+2\eta}\times\frac{e}{f}\mathrm{C} = -\frac{\theta\,.\,\theta+\eta\,.\,\theta+2\eta}{s\,.\,s+\eta\,.\,s+2\eta}\times\frac{e^3}{f^3}\mathrm{A}.$$

And in general the fluent of $\dot{\mathrm{A}}x^{m\eta}$ will be

$$\frac{\theta\,.\,\theta+\eta\,.\,\theta+2\eta\,\ldots\,\times\theta+m\eta-\eta}{s\,.\,s+\eta\,.\,s+2\eta\,\ldots\,\times s+m\eta-\eta}\times\frac{e^m}{f^m}\times\mathrm{A}.$$

If m is an even number the sign of the fluent will be $+$, but if it is an odd number the sign will be $-$.

Corol. 4. If $\lambda = 0$ or if $\dot{\mathrm{A}} = \frac{\dot{x}x^{\theta-1}}{e+fx^{\eta}}$ then $s = \theta$, and the fluent of $\dot{\mathrm{A}}x^{m\eta}$ is $\frac{e^m}{f^m}\times\mathrm{A}$.

Corol. 5. If $\dot{\mathrm{A}} = \frac{\dot{x}}{\sqrt{1-xx}}$ then $\theta = 1$, $\eta = 2$,

$\lambda =$

$\lambda = \frac{1}{2}$, $e = 1$, $f = -1$, and $s = 2$, put $\dot{N} = \dot{A}x^{2m}$, then $N = \frac{1 \cdot 3 \cdot 5 \ldots \times 2m - 1}{2 \cdot 4 \cdot 6 \ldots \times 2m} \times A$

and $\frac{N}{A} = \frac{1 \cdot 3 \cdot 5 \ldots \times 2m - 1}{2 \cdot 4 \cdot 6 \ldots \times 2m}$.

Scholium I.

199. By this proposition if one of the fluents A, B, C, &c. be given the rest may be found. Let $\dot{A} = \dot{x}x^{-\frac{1}{2}\eta - 1}\sqrt{e + fx^{\eta}}$, $\dot{B} = \dot{x}x^{\frac{1}{2}\eta - 1}\sqrt{e + fx_{\eta}}$, $\dot{C} = \dot{x}x^{\frac{3}{2}\eta - 1}\sqrt{e + fx^{\eta}}$, then $\theta = -\frac{1}{2}\eta$, $\lambda = \frac{3}{2}$ and $s = \eta$ and by Corol. 2. $-\frac{1}{2}\eta eA + \eta fB = x^{-\frac{1}{2}\eta} z^{\frac{3}{2}}$, and $\frac{1}{2}\eta eB + 2\eta f C = x^{\frac{1}{2}\eta} z^{\frac{3}{2}}$, or if we put $N^2 = \frac{e + fx_{\eta}}{x^{\eta}} = \frac{z}{x^{\eta}}$ or $z = N^2 x^{\eta}$, then $-\frac{1}{2}\eta eA + \eta f B = x^{\eta}N^3$, and $\frac{1}{2}\eta eB + 2\eta fC = x^{2\eta}N^3$, whence $B = \frac{x^{\eta}N^3}{\eta f} + \frac{eA}{2f}$; but by art. 94. $A = -\frac{2}{\eta}N + \frac{2}{\eta} R \left| \frac{R+T}{S} \right.$ therefore $B = \frac{x^{\eta}N^3}{\eta f} - \frac{eN}{\eta f} + \frac{e}{\eta f} R \left| \frac{R+T}{S} \right. = \frac{1}{\eta} x^{\eta} N + \frac{e}{\eta f} R \left| \frac{R+T}{S} \right.$.

In the same manner $C = \frac{1}{2\eta f} x^{2\eta} N^3 - \frac{e}{4f} B = \frac{1}{2\eta f} x^{2\eta} N^3 - \frac{e}{4\eta f} x^{\eta} N - \frac{e^2}{4\eta ff} R \left| \frac{R+T}{S} \right. = \frac{x^{\eta}N}{4\eta f} \times \overline{2N^2x^{\eta} - e} - \frac{e^2}{4\eta ff} R \left| \frac{R+T}{S} \right.$.

 Scholium

Scholium II.

We may also find the fluent of some fluxions by means of another fluent and an infinite series.

Example.

200. Let $\dot{A} = \dot{x}x^{\theta-1} \times \overline{e+fx^n}|^{\lambda-1}$, $\dot{B} = \dot{A}x^n$, $\dot{C} = \dot{B}x^n = \dot{A}x^{2n}$, and let it be required to find the fluent of $\dot{A} \times \overline{g+bx^n}|^v, = \dot{N}$, by expanding this binomial we have $\dot{N} = \dot{A}g^v + vg^{v-1}bx^n\dot{A} + \frac{v\,.\,v-1}{2} \times g^{v-2}b^2x^{2n}\dot{A} + \frac{v\,.\,v-1\,.\,v-2}{2\,.\,3} \times g^{v-3}b^3 x^{3n}\dot{A}$, &c. $= \dot{A}g^v + vg^{v-1}b\dot{B} + \frac{v\,.\,v-1}{2} \times g^{v-2}b^2\dot{C} + \frac{v\,.\,v-1\,.\,v-2}{2\,.\,3} \times g^{v-3}b^3\dot{D}$, &c. and taking the fluents $N = Ag^v + vg^{v-1}bB + \frac{v\,.\,v-1}{2} \times g^{v-2}b^2C + \frac{v\,.\,v-1\,.\,v-2}{2\,.\,3} \times g^{v-3}b^3D +$ &c. Where if A is known, the fluents B, C, D, &c. are found by Prop. XXXVIII. Corol. 2. and consequently the fluent N may be found.

201. To find the fluent N which is generated whilst x^n flows from o to $\frac{-e}{f}$, put $\theta + \lambda\eta = s$ and by Cor. 3. we have $B = -\frac{\theta e}{sf} \times A$, $C = + \frac{\theta\,.\,\theta+\eta}{s\,.\,s+\eta} \times \frac{e^2}{f^2} \times A$, $D = -\frac{\theta\,.\,\theta+\eta\,.\,\theta+2\eta}{s\,.\,s+\eta\,.\,s+2\eta} \times$

$\frac{e^s}{f^s} \times A$, and $N = Ag^v - \frac{v.\theta}{s} \times \frac{be}{f} \times g^{v-1} A + \frac{v.\overline{v-1}.\theta.\overline{\theta+\eta}}{2.s.\overline{s+\eta}} \times \frac{b^2e^2}{f^2} \times g^{v-2} A - \&c. = Ag^v$ into $1 - \frac{v.\theta}{s} \times \frac{be}{fg} + \frac{v.\overline{v-1}.\theta.\overline{\theta+\eta}}{2.s.\overline{s+\eta}} \times \frac{b^2e^2}{f^2g^2} -$ &c.

Corol. 1. If $g = e$ then $N = A \times e^v$ into $1 - \frac{v.\theta}{s} \times \frac{b}{f} + \frac{v.\overline{v-1}.\theta.\overline{\theta+\eta}}{2.s.\overline{s+\eta}} \times \frac{b^2}{f^2} -$ &c.

Corol. 2. If $\dot{A} = \dot{x}\sqrt{aa - xx}$ and $\dot{N} = \dot{A} \times \sqrt{aa - bxx}$ then $\theta = 1$, $\eta = 2$, $\lambda = \frac{1}{2}$, $v = \frac{1}{2}$, $s=4$, $e = g = a^2$, $f = -1$, $b = -b$ and $N = Aa \times 1 - \frac{1.1.b}{2.4} - \frac{1.1.1.3.b^2}{2.2.2.4.6} - \frac{1.1.3.1.3.5.b^3}{2.2.2.2.3.4.6.8} -$ &c. $= Aa \times 1 - \frac{b}{8} - \frac{b^2}{64} - \frac{5b^3}{1024} -$ &c.

Corol. 3. If $\dot{A} = \frac{\dot{x}}{\sqrt{a^2 - x^2}}$, and $\dot{N} = \dot{A} \times \sqrt{a^2 - bx^2}$ then $\theta = 1$, $\eta = 2$, $\lambda = \frac{1}{2}$, $v = \frac{1}{2}$, $s=2$, $e = g = a^2$, $f = -1$, $b = -b$, and $N = Aa \times \overline{1 - \frac{1.1.b}{2.2} - \frac{1.1.1.3.b^2}{2.2.2.2.4} - \frac{1.1.3.1.3.5.b^3}{2.2.2.2.3.2.4.6}} -$ &c. $= Aa \times \overline{1 - \frac{b}{4} - \frac{3b^2}{64} - \frac{5b^3}{256}}$ &c.

Corol. 4. Let $\dot{A} = \frac{v^2\dot{v}}{\sqrt{r^2 - v^2}}$ and $\dot{N} = \frac{\dot{A}v^2}{mr^2 + v^2} = \frac{\dot{B}}{mr^2 + v^2}$ where $\dot{B} = \frac{v^4\dot{v}}{\sqrt{r^2 - v^2}}$ then $\theta = 5$, $\eta = 2$, $\lambda = \frac{1}{2}$, $s = 6$, $\nu = -1$, $e = r^2$, $f = -1$, $g = mr^2$, $h = 1$, and $N = \frac{B}{mr^2}$ into $\overline{1 - \frac{1 \cdot 5}{6m} + \frac{1 \cdot 2 \cdot 5 \cdot 7}{2 \cdot 6 \cdot 8\, m^2} - \frac{1 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 3 \cdot 6 \cdot 8 \cdot 10\, m^3}}$ &c. $= \frac{B}{mr^2} \times \overline{1 - \frac{5}{6m} + \frac{5 \cdot 7}{6 \cdot 8 m^2} - \frac{5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10 m^3}}$ &c. But by Prop. XXXVIII. Corol. 3. $B = \frac{3r^2}{4} \times A$, therefore $N = A \times \frac{3}{4m} \times \overline{1 - \frac{5}{6m} + \frac{5 \cdot 7}{6 \cdot 8m^2} - \frac{5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10m^3}}$, &c. $= A \times \overline{\frac{3}{4m} - \frac{5}{8m^2} + \frac{5 \cdot 7}{8 \cdot 8m^3} - \frac{5 \cdot 7 \cdot 9}{8 \cdot 8 \cdot 10m^4}}$ &c.

PROP. XXXIX.

202. *Let* AOP *(fig.* 58.) *be any curve,* AO *the abscissa,* OP *an ordinate, I say the fluxion of the area* AOP *is equal to the fluxion of the abscissa multiplied into the ordinate.*

Call AO, x, OP, y, and at the point A, erect the perpendicular AL $=$ a given quantity a, and draw

draw LK parallel to AO, draw two ordinates op, $\omega\pi$ at equal infinitely small distances each side of OP, meeting KL in k and κ. Then since $Oo = O\omega$ the trapezium $op\pi\omega = o\omega \times OP$, and the rectangle $ok\kappa\omega = o\omega \times OK$ therefore $OK : OP :: ok\kappa\omega : op\pi\omega ::$ the fluxion of ALOK : the fluxion of AOP (art. 6.) and the fluxion of AOP $= \frac{OP}{OK} \times \overline{AL \cdot OK}$, but since ALOK $= ax$, the fluxion of ALOK $= a\dot{x}$ and the fluxion of AOP $= \frac{y}{a} \times a\dot{x} = y\dot{x}$. Q. E. D.

Corol. 1. If the rectangle AOPM is completed the fluxion of APM $= \dot{y}x$.

Corol. 2. If AP, $A\Pi$ (fig. 50.) be two curves having the same abscissa AO, and their ordinates are in a given ratio, then the areas will be in the same ratio.

Corol. 3. If in any two curves the ordinates are reciprocally as the fluxions of the abscissa's, the areas will be equal.

Therefore to find the area AOP, find the fluent of $\dot{x}y$ which must vanish when the ordinate passes through A, if the area between the ordinates at A and O is required.

Example I. *Fig.* 5.

203. *To find the area of the Apollonian parabola* AOP *contained between the vertex* A *and any ordinate* OP.

Call AO, x, OP, y, and let the latus rectum $= a$ then $ax = y^2$ and $a\dot{x} = 2y\dot{y}$ therefore the fluxion of the area $\dot{x}y = \frac{2y^2\dot{y}}{a}$ the fluent of which is $\frac{2y^3}{3a} + C$ which is to vanish when $y = 0$, therefore $C = 0$ and the area $AOP = \frac{2y^3}{3a} = \frac{2}{3}xy = \frac{2}{3}AO \times OP = \frac{2}{3}$ of the parallelogram AOPM.

Example II. *Fig*. 5. 13.

204. *To find the area of a parabola or hyperbola whose equation is* $y = ax^m$.

By multiplying this equation by $\dot{x}$ we have $\dot{x}y = a\dot{x}x^m$ whose fluent is $\frac{ax^{m+1}}{m+1} + C = \frac{xy}{m+1} + C$.

In a parabola, this area is equal to 0 when $x = 0$, and $C = 0$, therefore $AOP = \frac{xy}{m+1}$, and the area AOP is to the parallelogram AOMP as 1 to $m+1$.

In an hyperbola if $-m$ is less than 1, the fluent will $= 0$ when $x = 0$, and therefore the area contained between the ordinate OP and asymptote AG,

AG, infinite in height is equal to $\frac{AO \times OP}{m+1}$, and the other part OPH is infinite.

If $-m$ is greater than 1, then the area AOPGA will be infinite, but the other part will vanish when x is infinite and $C = 0$ therefore $OPH = \frac{AO \times OP}{m+1}$.

If $m = -1$ as in the common hyperbola, then both the parts AOPG, OPH will be infinite.

Example III. *Fig.* 59.

205. *To measure the area* BEOP *contained between two ordinates* BE, OP, *of the common hyperbola.*

Call AB, b, BO, x, OP, y, and let $AO \times OP = a$, then $y = \frac{a}{b+x}$ and $y\dot{x} = \frac{a\dot{x}}{b+x}$, the fluent of which by Prop. XIV. is $a \left| \frac{b+x}{b} \right.$ which vanishes when $x=0$ because $a \left| \frac{b}{b} \right. = 0$, therefore the area BEOP is equal to the measure of the ratio between AO and AB or between BE and OP, the modulus being a or the rectangle $AB \times BE$ *.

Or if M is the modulus of the system of logarithms the area $BEOP = \frac{a}{M} \times \log. \frac{b+x}{b}$.

* Cotes Harmon. Mensur. Prop. IV.

Take

Take QB = BO on the other side of B, then the area QREB $= -a\,\Big|\frac{b-x}{b} = a\,\Big|\frac{b}{b-x}$ and the whole area QRPO $= a\,\Big|\frac{b}{b-x} + a\,\Big|\frac{b+x}{b} =$

$a\,\Big|\frac{b+x}{b-x}$ which by art. 81 $= 2a \times$

$\frac{x}{b} + \frac{x^3}{3b^3} + \frac{x^5}{5b^5}$, &c.

Example IV. *Fig.* 1.

206. *To find the area of a circle.*

Call the diameter AB, a, the versed sine AO, x, the right sine OP, y, then by the nature of the circle $y^2 = ax - xx$ and the fluxion of the area AOP $= \dot{x}y = \dot{x}\sqrt{ax-xx}$ which reduced into an infinite series is equal $\dot{x} \times a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16a^{\frac{5}{2}}}$ &c. whose fluent is $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{5a^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{28a^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{72a^{\frac{5}{2}}}$ &c. which is equal to the area AOP.

To find the area of the sector ACP, the triangle COP $= \frac{1}{2}$ CO $\times$ OP $= \frac{1}{2}y \times \overline{\frac{1}{2}a - x} = \overline{\frac{1}{4}a - \frac{1}{2}x} \times \sqrt{ax - xx} = \frac{1}{4}a^{\frac{3}{2}}x^{\frac{1}{2}} - \frac{5}{8}a^{\frac{1}{2}}x^{\frac{3}{2}} + \frac{7x^{\frac{5}{2}}}{32a^{\frac{1}{2}}} + \frac{3x^{\frac{7}{2}}}{64a^{\frac{3}{2}}} +$ &c. which added to the above

value

value of AOP gives $ACP = \frac{1}{4} a^{\frac{3}{2}} x^{\frac{1}{2}} + \frac{1}{24} a^{\frac{1}{2}} x^{\frac{3}{2}} + \frac{3x^{\frac{5}{2}}}{160a^{\frac{1}{2}}} + \frac{5x^{\frac{7}{2}}}{448a^{\frac{3}{2}}} +$ &c.

Or if we put A for the first term, B for the second, and so on, then $ACP = \frac{1}{4} a^{\frac{3}{2}} x^{\frac{1}{2}} + \frac{1 \cdot 1}{2 \cdot 3} A \frac{x}{a} + \frac{3 \cdot 3}{4 \cdot 5} B \frac{x}{a} +$ &c.

Let the diameter $AB = a = 1$, and let the arc $AP = 60$ degrees, then $x = \frac{1}{4}$ and the sector $ACP = \frac{1}{4 \cdot 2} + \frac{1}{6 \cdot 4} A + \frac{9}{20 \cdot 4} B$ &c. and adding these terms as in the margin, the sector $ACP = 0.1308996 +$ and consequently the whole circle $= 6 \times ACP = 0.7853976 +$.

A =	0.1250000
B =	52083
C =	5859
D =	872
E =	148
F =	27
G =	5
H =	2
	0.1308996

Corol. 1. Since the area of a circle is equal to $\frac{1}{4}$ of the diameter multiplied into the circumference; The circumference of a circle whose diameter is 1 will be $4 \times 0.785398 = 3.141592$.

Corol. 2. Since the sector $ACP = \frac{1}{2} CP \times AP$; the arc AP will be $= \frac{2 \times ACP}{CP} = \frac{4 \times ACP}{a} =$

$$a^{\frac{1}{2}} x^{\frac{1}{2}} + \frac{1 \cdot 1}{2 \cdot 3} A \frac{x}{a} + \frac{3 \cdot 3}{4 \cdot 5} B \frac{x}{a} + \text{\&c.}$$

Corol.

Corol. 3. If we put CA$=r$, and CO$=z=\frac{a}{2}-x=r-x$ then $y^2=rr-zz$ and the area COPD$=$ the fluent of $\dot{z}\sqrt{rr-zz}$, or since $y\dot{y}=-z\dot{z}$, and $y\dot{z}=\frac{y^2\dot{y}}{z}$, COPD $=$ the fluent of $\frac{y^2\dot{y}}{\sqrt{rr-yy}}$.

Corol. 4. Since the triangle COP $=\frac{1}{2}zy$ we have $\overline{\dot{C}\dot{O}\dot{P}}=\frac{1}{2}\dot{z}y+\frac{1}{2}z\dot{y}=\frac{1}{2}\dot{z}y-\frac{z^2\dot{z}}{2y}$ and $\overline{\dot{C}\dot{P}\dot{D}}=\overline{\dot{C}\dot{O}\dot{P}\dot{D}}-\overline{\dot{C}\dot{O}\dot{P}}=y\dot{z}-\frac{1}{2}y\dot{z}+\frac{z^2\dot{z}}{2y}=\frac{\dot{z}}{2y}\times\overline{y^2+z^2}=\frac{r^2\dot{z}}{2y}=\frac{r^2\dot{z}}{2\sqrt{r^2-z^2}}$ therefore the sector CPD is equal to the fluent of $\frac{r^2\dot{z}}{2\sqrt{r^2-z^2}}=\frac{r^2\dot{y}}{2\sqrt{r^2-y^2}}$.

Corol. 5. In the same manner the fluxion of the sector ACP $=\overline{\dot{A}\dot{O}\dot{P}}+\overline{\dot{P}\dot{O}\dot{C}}=y\dot{x}+\frac{1}{2}y\dot{z}+\frac{1}{2}\dot{y}z$. But $\dot{z}=-\dot{x}$, and $\dot{y}=\frac{r\dot{x}-x\dot{x}}{y}=\frac{z\dot{x}}{y}$ therefore $\overline{\dot{A}\dot{C}\dot{P}}=y\dot{x}-\frac{1}{2}y\dot{x}+\frac{z^2\dot{x}}{2y}=\frac{r^2\dot{x}}{2y}=\frac{r^2\dot{x}}{2\sqrt{ax-xx}}=\frac{a^2\dot{x}}{8\sqrt{ax-xx}}$

Scholium.

Hence we may find the fluents of several fluxions by the help of circular areas.

Example

Example 1.

207. To find the fluent of $d\dot{z}z^{\frac{3n}{2}-1}\sqrt{e-fz^n}$ taking any invariable quantity g, put $e=ag$, $fz^n=gx$, then $nfz^{n-1}\dot{z}=g\dot{x}$ and $z^{n-1}\dot{z}=\frac{g\dot{x}}{nf}$. But since $z^n=\frac{gx}{f}$ we have $z^{\frac{1}{2}n}=\sqrt{\frac{gx}{f}}$ and therefore $d\dot{z}z^{\frac{3}{2}n-1}\sqrt{e-fz^n}=\frac{dg\dot{x}}{nf}\times\sqrt{\frac{gx}{f}}\times\sqrt{ag-gx}=\frac{dg^2}{nf^{\frac{3}{2}}}\times\dot{x}\sqrt{ax-xx}$. But by the last article it appears that the fluent of $\dot{x}\sqrt{ax-xx}$ is the area AOP, therefore the fluent of $d\dot{z}z^{\frac{3n}{2}-1}\sqrt{e-fz^n}$ is $\frac{dg^2}{nf^{\frac{3}{2}}}\times$ AOP, where AB $=\frac{e}{g}$ and AO $=\frac{fz^n}{g}$.

Example 2.

208. To find the fluent of $d\dot{z}z^{\frac{1}{2}n-1}\sqrt{e-fz^n}$. put $\theta=\frac{1}{2}n$, $\lambda=\frac{1}{2}$, $\dot{A}=\dot{z}z^{\frac{1}{2}n-1}\sqrt{e-fz^n}$, $\dot{B}=\dot{A}z^n=\dot{z}z^{\frac{3}{2}n-1}\sqrt{e-fz^n}$, then by Prop. XXXVIII. we have $\frac{1}{2}neA-2nfB=z^{\frac{1}{2}n}\times\overline{e-fz^n}|^{\frac{3}{2}}$ and $A=\frac{4fB}{e}+\frac{2}{ne}\times z^{\frac{1}{2}n}\times\overline{e-fz^n}|^{\frac{3}{2}}$, or using the same notation as in the last example we have $A=\frac{4fB}{e}+\frac{2}{nag}\times\sqrt{\frac{gx}{f}}\times\overline{ag-gx}|^{\frac{3}{2}}=\frac{4fB}{ag}+$

2g

$\frac{2g}{na\sqrt{f}} \times \overline{a-x} \times \sqrt{ax-xx}$ But $B = \frac{g^2}{nf^{\frac{3}{2}}} \times AOP$ therefore $A = \frac{4g}{na\sqrt{f}} \times AOP + \frac{2g}{na\sqrt{f}} \times OP \times BO = \frac{4g}{na\sqrt{f}} \times BPA$, therefore the fluent of $d\dot{z}z^{\frac{1}{2}n-1}\sqrt{e-fz^n}$ is equal to $\frac{4dg}{na\sqrt{f}} \times APB$.

Example 3.

209. To find the fluent of $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{\sqrt{e-fz^n}}$ put $e = ag$, $fz^n = gx$, then by art. 207. $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{\sqrt{e-fz^n}} = \frac{dg\dot{x}}{nf} \times \sqrt{\frac{f}{gx}} \times \frac{1}{\sqrt{ag-gx}} = \frac{8d}{na^2\sqrt{f}} \times \frac{a^2\dot{x}}{8\sqrt{ax-xx}}$. But the fluent of $\frac{a^2\dot{x}}{8\sqrt{ax-xx}}$ is equal to the sector CAP by art. 206. Corol. 5. therefore the fluent of $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{\sqrt{e-fz^n}}$ is equal to $\frac{8d}{na^2\sqrt{f}} \times CAP$.

Example 4.

210. To find the fluents of $\frac{d\dot{z}z^{\frac{3}{2}n-1}}{\sqrt{e-fz^n}}$ and $\frac{d\dot{z}z^{\frac{5}{2}n-1}}{\sqrt{e-fz^n}}$ put $\dot{A} = \frac{\dot{z}z^{\frac{1}{2}n-1}}{\sqrt{e-fz^n}}$, $\dot{B} = \dot{A}z^n$, $\dot{C} = \dot{B}z^n$, $\theta = \frac{1}{2}n$, $\lambda = \frac{1}{2}$ then by Prop. XXXVIII. we have

$\frac{1}{2}n$

$\frac{1}{2}\,neA - nfB = z^{\frac{1}{2}n}\sqrt{e-fz^n}$, and $\frac{3}{2}\,neB - 2nfC = z^{\frac{3n}{2}}\sqrt{e-fz^n}$ or $\frac{1}{2}\,nagA - nfB = \frac{g}{\sqrt{f}}\sqrt{ax-xx} = \frac{g}{\sqrt{f}} \times OP$, and $\frac{3}{2}\,nagB - 2nfC = \frac{g^2}{\sqrt{f^3}} \times x\sqrt{ax-xx} = \frac{g^2}{\sqrt{f^3}} \times AO \times OP$. Therefore $B = \frac{agA}{2f} - \frac{g}{n\sqrt{f^3}} \times OP$. But $A = \frac{8}{na^2\sqrt{f}} \times CAP$ and $\frac{agA}{2f} = \frac{4g}{na\sqrt{f^3}} \times CAP$ therefore $B = \frac{4g}{na\sqrt{f^3}} \times CAP - \frac{g}{n\sqrt{f^3}} \times OP = \frac{4g}{na\sqrt{f^3}} \times CAP - \frac{4g}{na\sqrt{f^3}} \times OP \times \frac{a}{4} = \frac{4g}{na\sqrt{f^3}} \times \overline{CAP - \Delta,\ ACP} = \frac{4g}{na\sqrt{f^3}} \times \overline{AOP - \Delta,\ AOP}$ that is the fluent of $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{\sqrt{e-fz^n}}$ is equal $\frac{4dg}{na\sqrt{f^3}} \times \overline{AOP - \Delta,\ AOP}$.

In the same manner $C = \frac{3agB}{4f} - \frac{g^2}{2n\sqrt{f^5}} \times AO \times OP = \frac{3g^2}{n\sqrt{f^5}} \times \overline{AOP - \Delta,\ AOP} - \frac{g^2}{n\sqrt{f^5}} \times \Delta,\ AOP = \frac{g^2}{n\sqrt{f^5}} \times \overline{3AOP - 4\Delta,\ AOP}$, and therefore the fluent of $\frac{d\dot{z}z^{\frac{5n}{2}-1}}{\sqrt{e-fz^n}} = \frac{dg^2}{n\sqrt{f^5}} \times \overline{3AOP - 4\Delta, AOP}$.

Example

Example V. *Fig.* 62.

211. *To find the area of an ellipse.*

Call the Latus rectum a, AB, $\frac{a}{b}$, AO, x, OP, y, then $y^2 = ax - bxx = \overline{\frac{a}{b}x - xx} \times b$ and $\dot{y}x = b^{\frac{1}{2}}\dot{x}\sqrt{\frac{a}{b}x - xx}$. On the diameter AB describe the circle AMB meeting OP produced in M, then by art. 206. the fluent of $\dot{x}\sqrt{\frac{a}{b}x - xx}$ is equal to AOM, and the fluent of $b^{\frac{1}{2}}\dot{x}\sqrt{\frac{a}{b}x - xx}$ or the elliptic area AOP is equal to $b^{\frac{1}{2}} \times$ AOM. Or as $\sqrt{b} = \sqrt{\frac{a}{AB}} = \frac{ED}{AB}$ we have AOP $= \frac{ED}{AB} \times$ AOM.

Corol. 1. The whole area of the ellipse $= \frac{ED}{AB}$ multiplied into the area of the circle AMB*m*.

Corol. 2. Since the areas of circles are as the squares of their diameters, the area of the ellipse will be as $\frac{ED}{AB} \times AB^2$, that is, as the rectangle under the axes ED $\times$ AB.

Corol. 3. Let D be the diameter of a circle whose area is equal to the ellipse, then DD $= \frac{ED}{AB}$

$\times AB^2$

$\times AB^2 = ED \times AB$, that is, D is a mean proportional between the two axes.

Corol. 4. The ellipse is a mean proportional between the inscribed and circumscribed circles. For the ellipse is to the circumscribed circle as ED : AB (by Corol. 1.) and $ED^2 : AB^2 ::$ inscribed circle : circumscribed circle, therefore the inscribed circle is to the ellipse as ED : AB, that is, as the ellipse to the circumscribed circle.

Example VI. *Fig.* 7.

212. *Let* AP *be an hyperbola, whose center is* S *and vertex* A, *to find the area* APO.

Let the two semiaxes be AG, and AS, call AS, a, AG, b, SO, x, OP, y, then $y^2 = \frac{b^2}{a^2} \times \overline{x^2 - a^2}$ and the fluxion of the area APO $= y\dot{x} = \frac{b\dot{x}}{a}\sqrt{xx - aa}$.

The sector SAP is equal to Δ, SOP — APO therefore $\overline{\text{SAP}} = \frac{1}{2}\dot{x}y + \frac{1}{2}x\dot{y} - \dot{x}y = \frac{1}{2}x\dot{y} - \frac{1}{2}\dot{x}y$, but since $\frac{a^2y^2}{b^2} = x^2 - a^2$ we have $\dot{y} = \frac{b^2x\dot{x}}{a^2y}$ and

$$\overline{\text{SAP}} = \frac{b^2x^2\dot{x}}{2a^2y} - \frac{\dot{x}y}{2} = \frac{1}{2}\dot{x} \times \frac{b^2x^2 - a^2y^2}{a^2y} =$$

$\frac{a^2b^2\dot{x}}{2a^2y} = \frac{b^2\dot{x}}{2y} = \frac{ab}{2} \times \frac{\dot{x}}{\sqrt{xx - aa}}$ which compared with the fluxion in Exam. 3. art. 94. gives $d =$

M ab

$\frac{ab}{2}$, $\eta = 2$, $e = -a^2$, $f = 1$ and $R = 1$, $T = \sqrt{\frac{xx - a^2}{xx}}$, $S = \sqrt{\frac{-a^2}{x^2}} = \frac{a}{x}\sqrt{-1}$ and the fluent $\frac{2}{\eta f} dR \left| \frac{R+T}{S} \right.$ is equal to $\frac{ab}{2} \left| \frac{1 + \frac{\sqrt{xx - aa}}{x}}{\frac{a}{x}\sqrt{-1}} \right. +$ $C = \frac{ab}{2} \left| \frac{\frac{bx}{a} + y}{b\sqrt{-1}} \right. + C$ which is to vanish when $x = a$, and $y = 0$ therefore $C = -\frac{ab}{2} \left| \frac{1}{\sqrt{-1}} \right.$ and the correct fluent is $\frac{ab}{2} \left| \frac{\frac{bx}{a} + y}{b} \right.$.

Let SL be a 4th proportional to SA, AG, and SO, then the sector $SAP = \frac{AS \times AG}{2} \left| \frac{SL + OP}{AG} \right.$. and $APO = SOP - SAP = \frac{SO \times OP}{2} - \frac{AS \times AG}{2} \left| \frac{SL + OP}{AG} \right.$ *. Q. E. I.

Scholium.

213. Hence we may find the fluents of some fluxions by means of hyperbolic areas: Thus the fluent of $\frac{b\dot{x}}{a}\sqrt{x^2 - a^2}$ is equal to the area AOP or the

* Cotes Harmon. Mensur. p. 25.

or the fluent of $d\dot{x}\sqrt{xx-aa}=\frac{da}{b}\times AOP$, and the fluent of $\frac{ab}{2}\times\frac{\dot{x}}{\sqrt{x^2-a^2}}$ is equal to the sector SAP, therefore the fluent of $\frac{d\dot{x}}{\sqrt{x^2-a^2}}=\frac{2d}{ab}\times$ SAP.

Let PT be a tangent in the point P, then $SO=\frac{SA^2}{ST}$, call ST, z, then $x=\frac{a^2}{z}$ and $\dot{x}=\frac{-a^2\dot{z}}{z^2}$, and $\sqrt{x^2-a^2}=\sqrt{\frac{a^4}{z^2}-a^2}=\frac{a}{z}\sqrt{a^2-z^2}$, therefore $\frac{b\dot{x}}{a}\sqrt{x^2-a^2}=\frac{a^2b\dot{z}}{z^3}\sqrt{a^2-z^2}$ therefore the fluent of $\frac{d\dot{z}}{z^3}\sqrt{a^2-z^2}$ is equal to $\frac{d}{a^2b}\times AOP$.

In the same manner $\frac{ab}{2}\times\frac{\dot{x}}{\sqrt{x^2-a^2}}=\frac{ab}{2}\times\frac{a^2\dot{z}}{z^2}\times\frac{z}{a\sqrt{a^2-z^2}}=-\frac{a^2b\dot{z}}{2z\sqrt{a^2-z^2}}$ and the fluent of $\frac{d\dot{z}}{z\sqrt{a^2-z^2}}=\frac{2d}{a^2b}\times$ SAP.

Example VII. *Fig.* 9.

214. *To find the area of the Cissoid of Diocles.*

Let AMB be the generating circle, G its center, BR the asymptote, OP an ordinate meeting the circle AMB in M; call AB, a, AO, x, OP, y; then $y^2=\frac{x^3}{a-x}$ and $y\dot{x}=\frac{\dot{x}x^{\frac{3}{2}}}{\sqrt{a-x}}$, which com-

pared with the fluxion $\dot{C}$ in art. 210 gives $d = 1$, $n = 1$, $e = a$, $f = 1$, $g = 1$ and the fluent $=$ $3AOM - 4\Delta$, AOM, therefore the area AOP is equal to $3AOM - 4\Delta$, $AOM = 3AKM - \Delta$, AOM. Q. E. I.

Corol. When O comes to B, the triangle AOM vanishes and the area APpRB infinite in length is equal to three times the semicircle AMB.

Example VIII. *Fig.* 63.

215. *To find the area of the* Conchoid *of* Nicomedes.

Let F be the pole and GQ the asymptote. Call the distance $AG = PR$, a, FG, b, GO, x, OP, y, then $y = \frac{b+x}{x}\sqrt{a^2 - x^2}$ and the fluxion of the area

$$AOP = -\dot{x}y = -\dot{x}\sqrt{a^2 - x^2} - \frac{b\dot{x}}{x}\sqrt{a^2 - x^2}.$$

With the center G and radius GA describe the circle AMK meeting OP in M, then by art. 206. Corol. 3. the fluent of the first part $-\dot{x}\sqrt{a^2 - x^2}$ which vanishes when $x = a$ is equal to the circular area AOM. The other part of the fluxion, $-\frac{b\dot{x}}{x}\sqrt{a^2 - x^2}$ may be compared with that in art. 94. Example V. which will give $d = -b$, $n = 2$, $e = a^2$, $f = -1$, $R = a$, $N = \sqrt{a^2 - x^2}$, $T = N$, $S = \sqrt{-x^2}$, $= x\sqrt{-1}$ and the fluent is

2d

$\frac{2d}{n}N - \frac{2d}{n}R\left|\frac{R+T}{S}\right. = -bN + ab\left|\frac{a+N}{x\sqrt{-1}}\right. +$ C which is to vanish when $x = a$, therefore $C = -ab\left|\frac{1}{\sqrt{-1}}\right.$, and the correct fluent is equal to $-bN + ab\left|\frac{a+N}{x}\right. = -FG \times RQ + GA \times FG\left|\frac{PR+RQ}{PQ}\right.$; which being added to the fluent of the first part $-x\sqrt{a^2-x^2}$ we have $AOP = AOM - FG \times RQ + GA \times FG\left|\frac{PR+RQ}{PQ}\right.$. But since FG : GR :: PQ : RQ, $FG \times RQ = GR \times PQ$ = the parallelogram PMGR therefore $AOP = AOM - PMGR + GA \times FG\left|\frac{PR+RQ}{PQ}\right.$ and adding OGRP to both sides of the equation $AGRP = AOM + OMG + GA \times FG\left|\frac{PR+RQ}{PQ}\right. = GAM + GA \times FG\left|\frac{PR+RQ}{PQ}\right.$.

Example IX. *Fig.* 15.

216. *To find the area of the logarithmic curve.*

Let AV be the first ordinate, and let PT be a tangent in the point P, call the subtangent OT, a, AO, x, OP, y, then $a = \frac{y\dot{x}}{\dot{y}}$ and $y\dot{x} = a\dot{y}$ whose fluent is $ay = OT \times OP$, therefore the infinitely long

long area OPB, contained between the curve and asymptote AB is equal OT $\times$ OP.

For the same reason AVB $=$ OT $\times$ AV and AVPO $= \overline{AV - OP} \times OT =$ (If PF is parallel to AO) VF $\times$ OT.

Example X. *Fig.* 17.

217. *To find the area of a Cycloid:*

Let AL be the generating circle, V the vertex, H the point which describes the cycloid, call KO, z, and using the same notation as in art. 117, we have $x = a - z$ and $\dot{x} = -\dot{z}$, but by art. 117. $\frac{\dot{x}}{\dot{y}} = \frac{v}{b+x}$ whence $\frac{-\dot{z}}{\dot{y}} = \frac{v}{a+b-z}$ and $-\dot{y} = \frac{\dot{z}}{v} \times \overline{a+b-z}$, now because VPS $=$ VPOK $-$ PSKO we have $\dot{\overline{VPS}} = \dot{\overline{VPOK}} - \dot{\overline{PSKO}} = \dot{z}y - \dot{z}y - z\dot{y} = -z\dot{y} =$ $\frac{z\dot{z} \times \overline{a+b-z}}{v}$ put $a+b = c$ then $\dot{\overline{VPS}} =$ $\frac{c\dot{z}z^{\frac{1}{2}}}{\sqrt{2a-z}} - \frac{\dot{z}z^{\frac{3}{2}}}{\sqrt{2a-z}}$ which may be compared with the fluxions $\dot{B}$ and $\dot{C}$ in art. 210. where $n=1$, $e=2a$, $f=1$, $a=2a$, $g=1$, and in the first part of the fluxion $d=c$ and the fluent is $\frac{2c}{a} \times$ $\overline{KOm - \Delta, KOm}$. In the second part $d = 1$, and the fluent is $3KOm - 4\Delta$, KOm and the area VPS is equal to the difference of the fluents

2c—

$\frac{2c-3a}{a} \times KOm - \frac{2c-4a}{a} \times \Delta, KOm =$

$\frac{2b-a}{a} \times KOm - \frac{2b-2a}{a} \times \Delta, KOm$. Q. E. I.

Corol. 1. When P comes to H, the area VHK will equal to $\frac{2b-a}{a} \times$ the semicircle KmH.

Corol. 2. In the common cycloid $a=b$, the area VPS $=$ KOm, and VHK equal to the semicircle KmH.

Example XI. *Fig.* 25.

218. *To find the area of the quadratrix.*

Call AG, r, GO, x, OP, y, EF, v, then OP $=$ AF and $\dot{y} = \dot{AF}$. Now it appears from art. 206. Corol. 4. that the fluxion of AF $=$ $\frac{r\dot{v}}{\sqrt{r^2-v^2}}$ the fluent of which by art. 47. is equal $v +$ $\frac{v^3}{6r^2} + \frac{3v^5}{40r^4} + \frac{5v^7}{112r^6}$ &c. which is equal to y; therefore by reversing this series by art. 73. we have $v = y - \frac{y^3}{6r^2} + \frac{y^5}{120r^4} - \frac{y^7}{5040r^6}$ &c. and $GE = \sqrt{r^2-v^2} = r - \frac{y^2}{2r} + \frac{y^4}{24r^3} - \frac{y^6}{720r^5}$ &c. whence $\frac{GE}{FE} = \frac{r}{y} - \frac{y}{3r} - \frac{y^3}{45r^3} -$ $\frac{2y^5}{945r^5}$ &c. and GO or $x = \frac{GE \times OP}{EF} =$

 $r-$

$r - \frac{y^2}{3r} - \frac{y^4}{45r^3} - \frac{2y^6}{945r^5}$ &c. therefore $\dot{x} =$ $-\frac{2y\dot{y}}{3r} - \frac{4y^3\dot{y}}{45r^3} - \frac{4y^5\dot{y}}{315r^5}$ &c. and $-y\dot{x} =$ $\frac{2y^2\dot{y}}{3r} + \frac{4y^4\dot{y}}{45r^3} + \frac{4y^6\dot{y}}{315r^5}$ &c. whose fluent $\frac{2y^3}{9r} +$ $\frac{4y^5}{9 \cdot 25r^3} + \frac{4 \cdot 5y^7}{9 \cdot 25 \cdot 49r^5}$ &c. is equal to the area AOP. Q. E. I.

Example XII. *Fig.* 60.

219. Let NAn be a semicircle, and in the right sine ZA take eZ : AZ :: AZ^2 : $m \times$ AT^2 + AZ^2, to find the area of the figure NFn which is the locus of the point e *.

Call NT, r, TZ, x, AZ, v, then $eZ = \frac{v^3}{mr^2 + v^2}$ and the fluxion of the area eZTF $=$ $eZ \times \dot{x} = \frac{v^3\dot{x}}{mr^2 + v^2} = \frac{v^4\dot{v}}{mr^2 + v^2} \times \sqrt{r^2 - v^2}$.

Let the circular area AZTB = A then by art. 206. Cor. 3. $\dot{A} = \frac{v^2\dot{v}}{\sqrt{r^2 - v^2}}$ and the fluxion of the area eZTF $= \frac{\dot{A}v^2}{mr^2 + v^2}$ whose fluent answering to the abscissa NT is equal (by art. 201. Corol. 4.) to $A \times \frac{3}{4m} - \frac{5}{8m^2} + \frac{5 \cdot 7}{8 \cdot 8m^3} - \frac{5 \cdot 7 \cdot 9}{8 \cdot 8 \cdot 10m^4}$ &c.

* See Newton Princip. lib. iii. prop. 32.

and

and the whole area NFn is to the area of the semicircle NAn as $\frac{3}{4m} - \frac{5}{8m^2} + \frac{5 \cdot 7}{8 \,.\, 8m^3} - \frac{5 \cdot 7 \cdot 9}{8 \,.\, 8 \,.\, 10m^4}$, &c. to 1.

Let $m = 9.08276$, then $\frac{1}{m} = .11010$, and

$$\begin{array}{ll|ll} +\frac{3}{4m} = & +.08257 & -\frac{5}{8m^2} = & -.00757 \\ +\frac{5 \cdot 7}{8 \,.\, 8m^3} = & .72 & -\frac{5 \cdot 7 \cdot 9}{8 \,.\, 8 \,.\, 10m^4} = & 7 \\ \hline & +.08329 & & -.00764 \\ & -\ \ \ 764 & & \\ \hline & .07565 & & \end{array}$$

therefore the area NFn is to the area of the semicircle as .07565 to 1 or as 60 to 793 nearly.

PROP. XL.

220. *To find the area of a curve expressed by an affected equation consisting of three terms, as* $y^\alpha \times e + f y^\eta x^\delta = k x^\beta$ *.

Put $\frac{\eta - \delta}{\eta} = s$ or $1 - \frac{\delta}{\eta} = s$, and $\frac{s\eta}{\alpha\delta + \beta\eta} = \lambda$, also put $z = \frac{1}{s} x^s$ and $v\dot{z} = y\dot{x}$ then $\dot{z} = \dot{x} x^{s-1}$ and $\frac{\dot{z}}{\dot{x}} = x^{s-1} = x^{\frac{-\delta}{\eta}}$ therefore $y = \frac{v\dot{z}}{\dot{x}} = v x^{\frac{-\delta}{\eta}}$ and $y^\eta x^\delta = v^\eta$, and $y^\alpha = v^\alpha x^{\frac{-\alpha\delta}{\eta}}$, by substituting

* Newton. Quadrat. Curvar. Prop. IX. Cor. 7.

these

these values in the equation of the curve we have, $v^a x^{\frac{-a\delta}{n}} \times \overline{e+fv^n} = kx^\beta$ or $v^a \times \overline{e+fv^n} = kx^{\frac{a\delta+n\beta}{n}} = kx^{\frac{s}{\lambda}}$, and therefore $v^{a\lambda}\times\overline{e+fv^n}|^\lambda = k^\lambda x^s$ and $x^s = v^{a\lambda} \times k^{-\lambda} \times \overline{e+fv^n}|^\lambda$ whence $z = \frac{1}{s}x^s = \frac{1}{s}v^{a\lambda} \times k^{-\lambda} \times \overline{e+fv^n}|^\lambda$ and $\dot{z} = \frac{\alpha\lambda}{s} \times \dot{v}v^{\alpha\lambda-1} \times k^{-\lambda} \times \overline{e+fv^n}|^\lambda + \frac{\lambda nf}{s} \times \dot{v}v^{\alpha\lambda+n-1} \times k^{-\lambda} \times \overline{e+fv^n}|^{\lambda-1}$ therefore $y\dot{x} = v\dot{z} = \frac{\alpha\lambda}{s} \times \dot{v}v^{\alpha\lambda} \times k^{-\lambda} \times \overline{e+fv^n}|^\lambda + \frac{\lambda nf}{s} \times \dot{v}v^{\alpha\lambda+n} \times k^{-\lambda} \times \overline{e+fv^n}|^{\lambda-1}$. The fluent of which will be the area of the curve proposed. Q. E. I.

The area may likewise be found a little easier thus, $\underline{|y\dot{x}|} = \underline{|v\dot{z}|} = vz - \underline{|\dot{v}z|}$, but $\dot{v}z = \frac{1}{s} \times \dot{v}v^{\alpha\lambda} \times k^{-\lambda} \times \overline{e+fv^n}|^\lambda$ which put equal $\dot{N}$ then $\underline{|y\dot{x}|} = vz - N$. Q. E. I.

Prop. XLI.

221. *Let* SWP (*fig.* 21.) *be a spiral, and retaining the same symbols as in Prop.* XXII. *I say the fluxion of the area* SWP *is equal to* $\frac{y^2\dot{x}}{2r}$.

For the triangle $SPp = \frac{1}{2}pn \times SP = \frac{Ff \times SP^2}{2SF}$ and $SFf = \frac{1}{2} Ff \times SF$ therefore $SF : \frac{SP^2}{SF} :: SFf : SPp ::$ the fluxion of SBF : the fluxion of SWP,

therefore

therefore the fluxion of SWP $= \frac{SP^2}{SF^2} \times \overline{\dot{SBF}}$, but SBF $= \frac{1}{2} rx$, and $\overline{\dot{SBF}} = \frac{1}{2} r\dot{x}$, therefore the fluxion of SWP is equal $\frac{y^2}{r^2} \times \frac{r\dot{x}}{2} = \frac{y^2\dot{x}}{2r}$.

Q. E. D.

222. Let $x : r :: y^m : a^m$, then by art. 122. $\dot{x} = \frac{mr\dot{y}y^{m-1}}{a^m}$ and $\frac{y^2\dot{x}}{2r} = \frac{m\dot{y}y^{m+1}}{2a^m}$ the fluent of which is $\frac{my^{m+2}}{2 . \overline{m+2} . a^m}$.

If $m+2$ is affirmative, then this fluent will vanish when $y = 0$, and the area SWP $= \frac{m}{2 \times \overline{m+2}} \times \frac{y^{m+2}}{a^m} = \frac{m}{m+2} \times \frac{y^2 x}{2r} = \frac{m}{m+2} \times \frac{y^2}{r^2} \times$ SBF.

If $m+2$ is negative, this fluent will vanish when y is infinite, and the area SPB (fig. 23.) contained between the asymptote SB and curve PB infinite in length is equal $-\frac{m}{2 . \overline{m+2}} \times \frac{y^{m+2}}{a^m}$.

If $m = -2$, then both the areas SWP and SPB will be infinite.

In the spiral of Archimedes $m = 1$, and the area SWP $= \frac{y^3}{6a}$.

In the reciprocal spiral, $m = -1$, and SWP $= \frac{1}{2} ay$.

If $m = -3$ then SPB $= \frac{3a^3}{2y}$.

223.

223. To find the area contained between two rays SW, SP of the lituus, put SW $= b$, and SP $= b + y$ then $x = \frac{a^2 r}{\overline{b+y}|^2}$ and $\dot{x} = -\frac{2a^2 r\dot{y}}{\overline{b+y}|^3}$ whence $\frac{\overline{b+y}|^2 \times \dot{x}}{2r} = \frac{a^2\dot{y}}{b+y}$ the fluent of which by Prop. XIV. is equal $a^2 \left| \frac{b+y}{b} \right.$. But since (by art. 122.) ST $= \frac{2a^2}{y}$, the triangle SPT will $= a^2$ and the area WSP is equal to the measure of the ratio of SP to SW the modulus being the triangle SPT *.

224. In the equiangular spiral $a\dot{y} = y\dot{x}$ (art. 124.) and therefore $\frac{y^2\dot{x}}{2r} = \frac{ay\dot{y}}{2r}$ the fluent of which is $\frac{ay^2}{4r}$ which vanishes when $y = 0$, therefore all the area from S to P is equal to $\frac{ay^2}{4r} = \frac{1}{4} y \times \frac{ay}{r} = \frac{1}{4}$ SP $\times$ ST $= \frac{1}{2}$ the triangle PST.

* Cotes Harm. Mensur. p. 86.

PROP.

PROP. XLII.

225. *Let* ADPd *(fig. 65.) be any curve,* OP *an ordinate to the base* ABO, *let* BO $=$ O$b = x$, OP $= y$, *then supposing the abscissa to flow uniformly, I say, the area* BDPO *generated on the abscissa* BO $= x$ *before the ordinate comes to* OP *is equal to* $yx - \frac{\dot{y}x^2}{2\dot{x}} + \frac{\ddot{y}x^3}{2 \cdot 3\dot{x}^2} - \frac{\dddot{y}x^4}{2 \cdot 3 \cdot 4\dot{x}^3}$ &c. *and the area* OPbd *on the abscissa* O$b = x$, *after the ordinate has passed* OP *is equal to* $yx + \frac{\dot{y}x^2}{2\dot{x}} + \frac{\ddot{y}x^3}{2 \cdot 3\dot{x}^2} + \frac{\dddot{y}x^4}{2 \cdot 3 \cdot 4\dot{x}^3}$ &c.

For let the area AOP $=$ A, then $\dot{A} = y\dot{x}$, $\ddot{A} = \dot{y}\dot{x}$, $\dddot{A} = \ddot{y}\dot{x}$, &c. and by Prop. XXV. the decrement of A whilst the decrement of the abscissa is x, is equal to $\frac{\dot{A}x}{\dot{x}} - \frac{\ddot{A}x^2}{2\dot{x}^2} + \frac{\dddot{A}x^3}{2 \cdot 3\dot{x}^3} -$ &c. $= yx - \frac{\dot{y}x^2}{2\dot{x}} + \frac{\ddot{y}x^3}{2 \cdot 3\dot{x}^2}$ &c. $=$ BDPO. And by the same proposition, the increment of A or OPbd is equal to $\frac{\dot{A}x}{\dot{x}} + \frac{\ddot{A}x^2}{2\dot{x}^2} + \frac{\dddot{A}x^3}{2 \cdot 3\dot{x}^3}$ &c. $= yx + \frac{\dot{y}x^2}{2\dot{x}} + \frac{\ddot{y}x^3}{2 \cdot 3\dot{x}^2}$ &c. Q. E. D.

Corol. The area BDdb is equal to $2yx + \ddot{y}x^3$

$$\frac{\frac{\ddot{y}x^3}{3\dot{x}^2} + \frac{\ddddot{y}x^5}{3\cdot 4\cdot 5\dot{x}^4} \text{ \&c.}}{y + \frac{\ddot{y}x^2}{2.3\dot{x}^2} + \frac{\ddddot{y}x^4}{2\cdot 3\cdot 4\cdot 5\dot{x}^4} \text{ \&c.}} = 2x \times$$

226. Call the ordinate *bd*, N, and let the difference of the first, third, fifth, &c. fluxions of OP, and *bd*, be *b*, *d*, *f*, &c. and let the difference of OP and *bd* be *a*, then the area OP*bd* is equal to $yx + \frac{ax}{2} - \frac{bx^2}{12\dot{x}} + \frac{dx^4}{720\dot{x}^3}$, &c. For suppose x given, then $a = N - y = \frac{\dot{y}x}{\dot{x}} + \frac{\ddot{y}x^2}{2\dot{x}^2} + \frac{\dddot{y}x^3}{6\dot{x}^3} + \frac{\ddddot{y}x^4}{24\dot{x}^4}$, &c. therefore $b = \dot{N} - \dot{y} = \frac{\ddot{y}x}{\dot{x}} + \frac{\dddot{y}x^2}{2\dot{x}^2} + \frac{\ddddot{y}x^3}{6\dot{x}^3}$, &c. and $d = \dddot{N} - \dddot{y} = \frac{\ddddot{y}x}{\dot{x}}$, &c. therefore adding equals to equals $yx + \frac{ax}{2} = yx + \frac{\dot{y}x^2}{2\dot{x}} + \frac{\ddot{y}x^3}{4\dot{x}^2} + \frac{\dddot{y}x^4}{12\dot{x}^3} + \frac{\ddddot{y}x^5}{48\dot{x}^4}$ &c. and adding $-\frac{bx^2}{12\dot{x}} = -\frac{\ddot{y}x^3}{12\dot{x}^2} - \frac{\dddot{y}x^4}{24\dot{x}^3} - \frac{\ddddot{y}x^5}{72\dot{x}^4}$, &c. we have $yx + \frac{ax}{2} - \frac{bx^2}{12\dot{x}} = yx + \frac{\dot{y}x^2}{2\dot{x}} + \frac{\ddot{y}x^3}{6\dot{x}^2} + \frac{\dddot{y}x^4}{24\dot{x}^3} + \frac{\ddddot{y}x^5}{144\dot{x}^4}$, &c. and again adding $+\frac{dx^4}{720\dot{x}^3} = \frac{\ddddot{y}x^5}{720\dot{x}^4}$ &c. we have $yx + \frac{ax}{2} - \frac{bx^2}{12\dot{x}} +$

dx^4

$\frac{dx^4}{720\dot{x}^3}$, &c. $= yx + \frac{\dot{y}x^2}{2\dot{x}} + \frac{\ddot{y}x^3}{6\dot{x}^2} + \frac{\dddot{y}x^4}{24\dot{x}^3} + \frac{\ddddot{y}x^5}{120\dot{x}^4}$ &c. $=$ OP*db*.

227. Let the base O*c* be divided into any number of equal parts as *n*, let the first ordinate OP$=y$ and the last $ce =$ N, and their difference, the difference of the first, third, fifth, &c. fluxions be equal *a*, *b*, *d*, &c. as before, let the sum of all the ordinates but the last $=$ S, then the area

$$\text{OP}ec = \frac{Sx}{n} + \frac{ax}{2n} - \frac{bx^2}{12n^2\dot{x}} + \frac{dx^4}{720n^4\dot{x}^3}, \text{ \&c.}$$

For call *bd*, K, *ob*, *z*, and the difference of *bd*, OP and the difference of their first, third, &c. fluxions, a, b, d, &c. then by the last article, the area OP*bd* $= yz + \frac{\text{a}z}{2} - \frac{\text{b}z^2}{12\dot{x}} + \frac{\text{d}z^4}{720\dot{x}^3}$ &c. In the same manner if $ce =$ L, and the difference of *bd* and *ce* the difference of their first, third, &c. fluxions be α, β, δ, &c. then $bdce = \text{K}z + \frac{\alpha z}{2} - \frac{\beta z^2}{12\dot{x}} + \frac{\delta z^4}{720\dot{x}^3}$ &c. And the sum of any number *n* of these areas or the area OP*ce* $=$ $\overline{y + \text{K} + \text{\&c.}} \times z + \cancel{\tfrac{1}{2}z \times \overline{\text{a} + \alpha + \text{\&c.}}} - \frac{z^2}{12\dot{x}} \times \overline{\text{b} + \beta + \text{\&c.}} + \frac{z^4}{720\dot{x}^3} \times \overline{\text{d} + \delta + \text{\&c.}} + \text{\&c.}$

but $z = \frac{x}{n}$, $y + \text{K} + \text{\&c.} = \text{S}$, $\text{a} + \alpha$ &c. $= a$, $\text{b} + \beta$ &c. $= b$ &c. therefore the area OP*ce* $=$ $\frac{Sx}{n} + \frac{ax}{2n} - \frac{bx^2}{12n^2\dot{x}} + \frac{dx^4}{720n^4\dot{x}^3}$, &c.

228.

228. Call the area OPce, P and put $\dot{x}=1$, then by the laſt article $P=\frac{Sx}{n}+\frac{ax}{2n}-\frac{bx^2}{12n^2}+\frac{dx^4}{720n^4}$, &c. but by art. 226. $P=yx+\frac{ax}{2}-\frac{bx^2}{12}+\frac{dx^4}{720}$, &c. or $\frac{P}{n^2}=\frac{yx}{n^2}+\frac{ax}{2n^2}-\frac{bx^2}{12n^2}+\frac{dx^4}{720n^2}$, &c. and the difference of theſe equations is $P\times\frac{n^2-1}{n^2}=\frac{\overline{S+\frac{1}{2}a}\times x}{n}-\frac{\overline{y+\frac{1}{2}a}\times x}{n^2}-\frac{\overline{n^2-1}\times dx^4}{720n^4}$, &c. therefore $P=\frac{\overline{S+\frac{1}{2}a}\times n-y-\frac{1}{2}a}{n^2-1}\times x-\frac{dx^4}{720n^2}$ &c. Let the ſum of the two extreme ordinates be equal A, the ſum of all the middle ordinates B, then $y+\frac{1}{2}a=\frac{1}{2}A$, $B=S-y$, and $S+\frac{1}{2}a=\frac{1}{2}A+B$. therefore $P=\frac{\frac{1}{2}A\times\overline{n-1}+Bn}{n^2-1}\times x-\frac{dx^4}{720n^2}$ &c. $=\frac{A.\overline{n-1}+2Bn}{2.\overline{n^2-1}}\times x-\frac{dx^4}{720n^2}$, &c.

If $n=2$, then B is the middle ordinate and the area is equal $\frac{A+4B}{6}\times x-\frac{dx^4}{4\cdot720}$, &c.

If $n=3$, then B is the ſum of the two middle ordinates, and the area $=\frac{A+3B}{8}\times x-\frac{dx^4}{9\cdot720}$, &c.

If

If there are five ordinates, let C be the middle ordinate, B the sum of the two ordinates next the midst, then $n=4$, and $P = \frac{3A + 8B + 8C}{30} \times x - \frac{dx^4}{16 . 720}$, &c. but by the rule for three ordinates $P = \frac{A + 4C}{6} \times x - \frac{dx^4}{4 . 720}$ or $\frac{1}{4} P = \frac{A + 4C}{24} \times x - \frac{dx^4}{16 . 720}$, the difference of these two equations is $\frac{3P}{4} = \frac{7A + 32B + 12C}{120} \times x$ and the area $P = \frac{7A + 32B + 12C}{90} \times x$.

229. If P, a, b, d, &c. be given, then S may be be found, for by art. 227. $\frac{Sx}{n} = P - \frac{ax}{2n} + \frac{bx^2}{12n^2} - \frac{dx^4}{720n^4}$, &c. and $S = \frac{Pn}{x} - \frac{a}{2} + \frac{bx}{12n} - \frac{dx^3}{720n^3}$, &c. let the distance of the ordinates $\frac{x}{n} = z$ then $S = \frac{P}{z} - \frac{a}{2} + \frac{bz}{12} - \frac{dz^3}{720}$ &c.

If $z = 1$, and the number of ordinates be infinite, then $a = y$, $b = \dot{y}$, $d = \dddot{y}$, &c. and $P = \boxed{y\dot{z}} = y'$ and $S = y' - \frac{1}{2} y + \frac{1}{12}\dot{y} - \frac{1}{720}\dddot{y}$ &c.

Hence any infinite series may be summed, as for example, let x be the distance from the beginning of the series and let $y = \frac{1}{x^n}$ then $y' =$

$\frac{-1}{\overline{n-1}\,.\,x^{n-1}}$, $\dot{y} = \frac{-n}{x^{n+1}}$, $\dddot{y} = -\frac{n.\overline{n+1}.\overline{n+2}}{x^{n+3}}$ &c.

and $S = \frac{1}{\overline{n-1}\,.\,x^{n-1}} + \frac{1}{2x^n} + \frac{n}{12x^{n+1}} - \frac{n.\overline{n+1}.\overline{n+2}}{720x^{n+3}}$ &c. $= \frac{1}{x^n} + \frac{1}{\overline{x+1}|^n} + \frac{1}{\overline{x+2}|^n}$ &c. *

If $x = 1$, $n = 2$ then $\frac{1}{1} + \frac{1}{4} + \frac{1}{9}$ &c. $= 1 + \frac{1}{2} + \frac{1}{6} - \frac{1}{30}$, &c.

If $n = -1$, then $y = -\log\frac{1}{x}$, $\dot{y} = -\frac{1}{x}$, $\dddot{y} = -\frac{6}{x^4}$. And $S = \log.\frac{1}{x} + \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4}$, &c. $= \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2}$, &c.

If $n = -1$, and $x = a$ then $\frac{1}{a} + \frac{1}{a+1} +$ &c. $= \log.\frac{1}{a} + \frac{1}{2a} + \frac{1}{12a^2} - \frac{1}{120a^4}$, &c.

which taken from the last leaves $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} \ldots\ldots \frac{1}{a-1} = \log.\frac{a}{x} + \frac{1}{2x} - \frac{1}{2a} + \frac{1}{12x^2} - \frac{1}{12a^2} - \frac{1}{120x^4} + \frac{1}{120a^4}$ &c.

* De Moivre Supplem. ad Miscell. Analyt. p. 19.

SECT. IX.

Of the Rectification of Curve Lines.

PROP. XLIII.

230. *If the abscissa* AO *(fig. 12.)* $= x$, *the ordinate* OP $= y$, *then the fluxion of the arc* AP *will be equal to* $\sqrt{\dot{x}^2 + \dot{y}^2}$.

Let PT be a tangent in the point P and let *op* be infinitely near OP, draw P*n* parallel to AO, and call AP, *t*, then OP : TP :: *np* : P*p* :: (art. 6.) $\dot{y} : \dot{t}$ and $\dot{t} = \frac{\dot{y}}{y} \times$ TP. But TP $=$

$$\sqrt{TO^2 + OP^2} = \sqrt{\frac{y^2\dot{x}^2}{\dot{y}^2} + y^2} = \frac{y}{\dot{y}}\sqrt{\dot{x}^2 + \dot{y}^2}$$

therefore $\dot{t} = \sqrt{\dot{x}^2 + \dot{y}^2}$. Q. E. D.

Corol. 1. If $\frac{\dot{y}}{\dot{x}} = z$ then $\dot{t} = \dot{x}\sqrt{1 + z^2}$.

Corol. 2. To find the length of any curve compute $\sqrt{\dot{x}^2 + \dot{y}^2}$ and its fluent will be the length of the curve.

Example I. *Fig.* 64.

231. *To find the length of a parabola.*

Call AO, x, OP, y, and let the latus rectum $= a$, then $yy = ax$, and $2y\dot{y} = a\dot{x}$, therefore $\dot{y} = \frac{a\dot{x}}{}$

N 2

$\frac{a\dot{x}}{2y}$ and $\dot{y}^2 = \frac{a^2\dot{x}^2}{4y^2} = \frac{a\dot{x}^2}{4x}$ whence $\dot{x}^2 + \dot{y}^2 = \dot{x}^2 \times \overline{1 + \frac{a}{4x}}$ and $\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x}x^{-\frac{1}{2}}\sqrt{\tfrac{1}{4}a + x}$ which may be compared with the fluxion in art. 94. Exam. VI. where $d = 1$, $n = 1$, $e = \frac{1}{4}a$, $f = 1$, $N = \sqrt{\frac{\frac{1}{4}a + x}{x}} = \sqrt{\frac{\frac{1}{4}ax + x^2}{x^2}} = \frac{1}{x}\sqrt{\tfrac{1}{4}y^2 + x^2}$, $R = 1$, $T = N$ and $S = \sqrt{\frac{a}{4x}} = \frac{y}{2x}$ and the fluent is

$$\frac{d}{n}x^n N + \frac{e}{nf}dR \left| \frac{R + T}{S} \right. = \sqrt{\tfrac{1}{4}y^2 + x^2} +$$

$$\tfrac{1}{4}a \left| \frac{1 + \frac{1}{x}\sqrt{\frac{1}{4}y^2 + x^2}}{\frac{y}{2x}} \right. = \sqrt{\tfrac{1}{4}y^2 + x^2} +$$

$$\tfrac{1}{4}a \left| \frac{x + \sqrt{\frac{1}{4}y^2 + x^2}}{\frac{1}{2}y} \right.$$. Let S be the focus, and let OP be bisected in L then $AP = AL + AS \left| \frac{AO + AL}{LO} \right.$ *.

Example II. *Fig.* 5.

232. *To find the length of a parabola whose equation is $y = ax^m$.*

By taking the fluxions $\dot{y} = max^{m-1}\dot{x}$, and $\dot{y}^2 = m^2a^2x^{2m-2}$ and $\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x}\sqrt{1 + m^2a^2x^{2m-2}}$ the

* Cotes Harmon. Mensur. p. 22.

fluent

fluent of which is equal to the arc AP.

If $m=\frac{3}{2}$, that is, if $y^2=a^2x^3$, then the fluxion of AP $=\dot{x}\sqrt{1+\frac{9}{4}a^2x}$. Put $1+\frac{9}{4}a^2x=N$ then $\dot{x}=\frac{4\dot{N}}{9a^2}$ and the fluxion of AP $=\frac{4\dot{N}N^{\frac{1}{2}}}{9a^2}$ whose fluent is $\frac{8N^{\frac{3}{2}}}{27a^2}+C$ which is to vanish when $x=0$ or $N=1$, therefore $C=-\frac{8}{27a^2}$ and AP $=\frac{8N^{\frac{3}{2}}-8}{27a^2}$.

If $m=\frac{4}{3}$ or $y^3=a^3x^4$, then the fluxion of AP is $\dot{x}\sqrt{1+\frac{16}{9}a^2x^{\frac{2}{3}}}$ which may be compared with art. 94. exam. 7. where $d=1$, $\eta=\frac{2}{3}$, $e=1$, $f=\frac{16a^2}{9}$, $N=\sqrt{\frac{1+\frac{16}{9}a^2x^{\frac{2}{3}}}{x^{\frac{2}{3}}}}=\frac{1}{3x^{\frac{1}{3}}}\sqrt{9+16a^2x^{\frac{2}{3}}}$, $R=\frac{4a}{3}$, $T=N$, $S=x^{-\frac{1}{3}}$ and the fluent is

$$\frac{e+2fx^{\eta}}{4\eta f}dx^{\eta}N-\frac{e^2}{4\eta ff}dR\left|\frac{R+T}{S}\right. =$$

$$\frac{1+\frac{32}{9}x^{\frac{2}{3}}}{\frac{128a^2}{27}}\times x^{\frac{2}{3}}N-\frac{1}{\frac{8}{3}\times\frac{256}{81}a^4}\times$$

$$\frac{4}{3}a\left|\frac{\frac{4a}{3}+\frac{1}{3x^{\frac{1}{3}}}\sqrt{9+16a^2x^{\frac{2}{3}}}}{x^{-\frac{1}{3}}}\right. =$$

N 3 9 +

$$\frac{9+32x^{\frac{2}{3}}}{128a^2} \times x^{\frac{1}{3}}\sqrt{9+16a^2x^{\frac{2}{3}}} -$$

$$\frac{81}{512a^3}\Bigg|\frac{4ax^{\frac{1}{3}}+\sqrt{9+16a^2x^{\frac{2}{3}}}}{3}.$$

Example III. *Fig.* 61.

233. *To find the length of a circular arc* AP *whose sine is* OP.

Call CO, x, OP, y, CA, r, AP, A, then $r^2 - x^2 = y^2$ and $-x\dot{x} = y\dot{y}$ whence $\dot{x}^2 = \frac{y^2\dot{y}^2}{x^2}$ and $\dot{x}^2+\dot{y}^2 = \frac{\dot{y}^2}{x^2} \times \overline{x^2+y^2} = \frac{r^2\dot{y}^2}{x^2}$ and $\dot{\text{A}} = \sqrt{\dot{x}^2+\dot{y}^2} = \frac{r\dot{y}}{\sqrt{r^2-y^2}}$ whose fluent (by art.47.) is $y + \frac{1.1}{2.3}\text{A}\frac{y^2}{r^2} + \frac{3.3}{4.5}\text{B}\frac{y^2}{r^2}$, &c. $=$ AP.

This fluent may be expressed in a different manner by art. 50. where $\theta = 1$, $e = r^2$, $f = -1$, $\eta = 2$, $\lambda = \frac{1}{2}$, $r = \frac{\theta+\eta}{\eta} = \frac{3}{2}$, $s = \frac{\theta+\lambda\eta}{\eta} = 1$ and the fluent is $\sqrt{r^2-y^2} \times \frac{y}{r} + \frac{2}{3}\text{A}\frac{y^2}{r^2} + \frac{4}{5}\text{B}\frac{y^2}{r^2}$ &c. $=$ AP. Q. E. I.

Corol. 1. Since $\text{A} = y + \frac{1}{6}\text{A}\frac{y^2}{r^2} + \frac{9}{20}\text{B}\frac{y^2}{r^2}$ &c. $= y + \frac{y^3}{6r^2} + \frac{9y^5}{120r^4}$ &c. by reversing this series

series by art 73. we have $y = A - \frac{A^3}{6r^2} + \frac{A^5}{120r^4}$ &c. $= A - \frac{A^3}{2.3r^2} + \frac{A^5}{2.3.4.5r^4} +$ &c.

Corol. 2. Since $x^2 + y^2 = r^2$ we have $x = \sqrt{r^2 - y^2} = \sqrt{r^2 - A^2 + \frac{A^4}{3r^2} - \frac{2A^6}{45r^4}}$, &c. $= r - \frac{A^2}{2r} + \frac{A^4}{24r^3} - \frac{A^6}{720r^5}$, &c. $= r - \frac{A^2}{2r} + \frac{A^4}{2.3.4r^3} - \frac{A^6}{2.3.4.5.6r^5}$, &c.

Corol. 3. Since $y\dot{y} = -x\dot{x}$ we have $\dot{A} = \frac{r\dot{y}}{x} = \frac{-r\dot{x}}{y} = \frac{-r\dot{x}}{\sqrt{r^2 - x^2}}$.

Corol. 4. Call the tangent AT, t, then $t = \frac{OP \times CA}{CO} = \frac{ry}{x} = \frac{ry}{\sqrt{r^2 - y^2}}$ and $t^2 = \frac{r^2y^2}{r^2 - y^2}$ or $\frac{1}{t^2} = \frac{1}{y^2} - \frac{1}{r^2}$ and $y^2 = \frac{r^2t^2}{r^2 + t^2}$, also taking the fluxions of the equation $\frac{1}{t^2} = \frac{1}{y^2} - \frac{1}{r^2}$ we have $\frac{\dot{t}}{t^3} = \frac{\dot{y}}{y^3}$ and $\dot{y} = \frac{\dot{t}y^3}{t^3} =$ (because $\frac{y}{t} = \frac{x}{r}$) $\frac{\dot{t}y^2x}{t^2r}$ therefore $\dot{A} = \frac{r\dot{y}}{x} = \frac{\dot{t}y^2}{t^2} = \frac{r^2\dot{t}}{r^2 + t^2} = \dot{t} - \frac{\dot{t}t^2}{r^2} + \frac{\dot{t}t^4}{r^4} - \frac{\dot{t}t^6}{r^6}$ &c. whose fluent $= A = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6}$, &c.

If the arc $A = 45^{\circ}$ and $r = 1$ then $t=1$, and $A = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ &c.

If we put the secant $= s$ then $\dot{A} = \frac{t r^2}{s^2}$ and taking the fluents by art. 56. $A = \frac{t}{s} + \frac{2}{3} A \frac{t^2}{s^2} + \frac{4}{5} B \frac{t^2}{s^2} +$ &c.

234. The fluxion $\frac{-r\dot{x}}{\sqrt{r^2 - x^2}}$ being compared with that in art. 94. exam. 3. we have $d = -r$, $\eta = 2$, $e = r^2$, $f = -1$, and $R = \sqrt{-1}$, $T = \sqrt{\frac{r^2 - x^2}{x^2}} = \frac{y}{x}$, $S = \frac{r}{x}$ and the fluent is

$$\frac{2}{\eta f} dR \left| \frac{R + T}{S} \right. = r\sqrt{-1} \left| \frac{x\sqrt{-1} + y}{r} \right. =$$

$CA\sqrt{-1} \left| \frac{CO\sqrt{-1} + OP}{CP} \right.$ which is equal to the arc AP *.

Let AT be a tangent at the point A, meeting OP produced in T, then since the triangles COP, CAT are similiar we have $\frac{CO\sqrt{-1} + OP}{CP} = \frac{CA\sqrt{-1} + AT}{CT}$ and the arc $AP =$

$CA\sqrt{-1} \left| \frac{CA\sqrt{-1} + AT}{CT} \right.$. Therefore the

* Cotes Harmon. Mensur. p. 28.

measure

measure $R \left| \frac{R+T}{S} \right.$ by changing the sign of R^2 becomes equal to the arc whose radius tangent and secant are $R\sqrt{-1}$, T, S. Thus the fluent of $\frac{\dot{x}x^{\frac{1}{2}n-1}}{\sqrt{e-fx^n}}$ by article 94. Example III. is equal $\frac{2}{nf} dR \left| \frac{R+T}{S} \right.$ where $R = \sqrt{-f}$ and $R\sqrt{-1} = \sqrt{f}$, $T = \sqrt{\frac{e-fx^n}{x^n}}$, $S = \sqrt{\frac{e}{x^n}}$ and therefore the fluent of $\frac{d\dot{x}x^{\frac{1}{2}n-1}}{\sqrt{e-fx^n}}$ is equal $\frac{2d}{nf} \times$ the arc whose radius, tangent and secant are $\sqrt{f}$, T and S or $\sqrt{f}$, $\sqrt{\frac{e-fx^n}{x^n}}$, $\sqrt{\frac{e}{x^n}}$ which is expressed by Mr. Cotes this $\frac{2d}{nf} R \left(\frac{R.T}{S} \right.$.

235. Let AP, and Ab (fig. 66.) be two arcs which are to each other as n to 1, call the radius r, the cosine of AP, z, the cosine of Ab, x, then by the last article $AP = r\sqrt{-1} \left| \frac{z\sqrt{-1}+\sqrt{r^2-z^2}}{r} \right.$ and $Ab = r\sqrt{-1} \left| \frac{x\sqrt{-1}+\sqrt{r^2-x^2}}{r} \right.$ but $AP = n \times Ab$ therefore $r\sqrt{-1} \left| \frac{z\sqrt{-1}+\sqrt{r^2-z^2}}{r} \right. = nr\sqrt{-1} \left| \frac{x\sqrt{-1}+\sqrt{r^2-x^2}}{r} \right.$ and

$z\sqrt{}$

$\frac{z\sqrt{-1}+\sqrt{r^2-z^2}}{r} = \frac{\overline{x\sqrt{-1}+\sqrt{r^2-x^2}}\big|^n}{r^n}$ and dividing by $\frac{\sqrt{-1}}{r}$, $z+\sqrt{z^2-r^2} = \frac{\overline{x+\sqrt{x^2-r^2}}\big|^n}{r^{n-1}}$. Put $x+\sqrt{x^2-r^2}=Q$ then $z+\sqrt{z^2-r^2}=\frac{Q^n}{r^{n-1}}$ and $\sqrt{z^2-r^2}=\frac{Q^n}{r^{n-1}}-z$ and squaring both sides we have $z^2-r^2 = \frac{Q^{2n}}{r^{2n-2}} - \frac{2zQ^n}{r^{n-1}} + z^2$ whence $Q^{2n}-2r^{n-1}zQ^n+r^{2n}=0$.

Also since $Q=x-\sqrt{x^2-r^2}$ we have $Q^2-2Qx+x^2=x^2-r^2$ or $Q^2-2xQ+r^2=0$. Therefore the relation of x to z may be found by exterminating Q out of the two equations $Q^{2n}-2r^{n-1}zQ^n+r^{2n}=0$, and $Q^2-2Qx+x^2=0$. and the quantity $Q^2-2xQ+r^2$ will be a divisor of $Q^{2n}-2r^{n-1}zQ^n+r^{2n}$.

236. We may also find the relation between x and z by means of an infinite series, for since $Q^{2n}-2r^{n-1}zQ^n+r^{2n}=0$, we have $2r^{n-1}z=Q^n+\frac{r^{2n}}{Q^n}$. But since $Q=x+\sqrt{x^2-r^2}$ we have $\frac{r^2}{Q}=x-\sqrt{x^2-r^2}$ and $2r^{n-1}z=$ $\overline{x+\sqrt{x^2-r^2}}\big|^n+\overline{x-\sqrt{x^2-r^2}}\big|^n$ put $r^2-x^2=y^2$ then $2r^{n-1}z=\overline{x+y\sqrt{-1}}\big|^n+\overline{x-y\sqrt{-1}}\big|^n$, but by

by Lemma II. we have $\overline{x+y\sqrt{-1}}|^n = x^n +$
$nx^{n-1}y\sqrt{-1} - \frac{n.\overline{n-1}}{2}x^{n-2}y^2 -$
$\frac{n.\overline{n-1}.\overline{n-2}}{2.3}x^{n-3}y^3\sqrt{-1} +$
$\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4}x^{n-4}y^4$, &c. and
$\overline{x-y\sqrt{-1}}|^n = x^n - nx^{n-1}y\sqrt{-1} -$
$\frac{n.\overline{n-1}}{2}x^{n-2}y^2 + \frac{n.\overline{n-1}.\overline{n-2}}{2.3}x^{n-3}y^3\sqrt{-1} +$
$\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4}x^{n-4}y^4$, &c. and $r^{n-1}z =$
$x^n - \frac{n.\overline{n-1}}{2}x^{n-2}y^2 +$
$\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4}x^{n-4}y^4$, &c.

237. If we extract the root of $x^2 - r^2$ by Lemma 2, we have $\sqrt{x^2 - r^2} = x - \frac{r^2}{2x} - \frac{r^4}{8x^3} - \frac{r^6}{16x^5}$, — &c. and $Q = x + \sqrt{x^2 - r^2} = 2x - \frac{r^2}{2x} - \frac{r^4}{8x^3} - \frac{r^6}{16x^5}$ &c. and by art. 45. $Q^n = 2^n x^n -$
$n.2^{n-2}x^{n-2}r^2 - n.2^{n-4}x^{n-4}r^4 - 2n.2^{n-6}x^{n-6}r^6$

$$+ \frac{n.\overline{n-1}}{2}.2^{n-4}x^{n-4}r^4 + \overline{n.n-1}.2^{n-6}x^{n-6}r^6$$

$$- \frac{n.\overline{n-1}.\overline{n-2}}{2.3}.2^{n-6}x^{n-6}r^6$$

$$= 2^n$$

$= 2^n . x^n - n . 2^{n-2} x^{n-2} r^2 + \frac{n.\overline{n-3}}{2} . 2^{n-4} x^{n-4} r^4 -$
$\frac{n.\overline{n-4}.\overline{n-5}}{2.3} . 2^{n-6} x^{n-6} r^6 +$ &c. also $\frac{r^2}{Q} =$
$\frac{r^2}{2x} + \frac{r^4}{8x^3}$, &c. and $\frac{r^{2n}}{Q^n} = \frac{r^{2n}}{2^n x^n}$, &c. therefore
$r^{n-1} z = 2^{n-1} . x^n - n . 2^{n-3} x^{n-2} r^2 +$
$\frac{n.\overline{n-3}}{2} . 2^{n-5} x^{n-4} r^4 - \frac{n.\overline{n-4}.\overline{n-5}}{2.3} \times$
$2^{n-7} x^{n-6} r^6$, &c.

If we put $r = 1$ then z or cos. $nA = 2^{n-1} x^n -$
$n . 2^{n-3} x^{n-2} + \frac{n.\overline{n-3}}{2} . 2^{n-5} x^{n-4}$, &c.

238. Hence we may find any power of x or cos. A by means of the cosines of the multiples of A. for by art. 236. If $r = 1$; cos. $nA = x^n -$
$\frac{n.\overline{n-1}}{2} x^{n-2} y^2 + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4} x^{n-4} y^4$, &c.
and in the same manner cos. $\overline{n-2} . A = x^{n-2} -$
$\frac{\overline{n-2}.\overline{n-3}}{2} x^{n-4} y^2$, &c. and cos. $\overline{n-4} . A =$
$x^{n-4} -$ &c. and adding equals to equals cos.
$\overline{n-2} . A = x^{n-2} - \frac{\overline{n-2}.\overline{n-3}}{2} \times y^2 \times$ cos.
$\overline{n-4} . A$. and cos. $nA = x^n - \frac{n.\overline{n-1}}{2} \times y^2 \times$
cos. $\overline{n-2} . A - \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.2} \times y^4 \times$

cos.

$\text{cos.}\,\overline{n-4}\,.\,A + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,3\,.\,4} \times y^4 \times$

$\text{cos.}\,\overline{n-4}.\,A\ \&c. = x^n - \frac{n.n-1}{2} \times y^2 \times \text{cos.}\,\overline{n-2}.$

$- \frac{5\,.\,n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,3\,.\,4}.\ y^4 \times \text{cos.}\,\overline{n-4}\,.\,A.$

&c. and therefore $x^n = \text{cos.}\,nA + \frac{n\,.\,n-1}{2}\,y^2 \times$

$\text{cos.}\,\overline{n-2}\,.\,A + \frac{5\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,3\,.\,4}\,.\,y^4 \times$

$\text{cos.}\,\overline{n-4}.\,A\ \&c.$

239. A similiar theorem follows from art. 237. for if $r=1$, $\text{cos.}\,nA = 2^{n-1}.\,x^n - n\,.\,2^{n-3}\,x^{n-2} + \frac{n.n-3}{2}.\,2^{n-5}\,x^{n-4}$ &c. $\text{cos.}\,\overline{n-2}.A = 2^{n-3}.x^{n-2} - \overline{n-2}.2^{n-5}\,x^{n-4}$, &c. $\text{cos.}\,\overline{n-4}\,.A = 2^{n-5}\,x^{n-4}$, &c. and adding equals to equals, $\text{cos.}\,\overline{n-2}.A = 2^{n-3}x^{n-2} - \overline{n-2} \times \text{cos.}\,\overline{n-4}\,.\,A$ &c. and $\text{cos.}\,nA = 2^{n-1}x^n - n \times \text{cos.}\,\overline{n-2}\,.\,A - \overline{n.n-2} \times \text{cos.}\,\overline{n-4}\,.\,A + \frac{n\,.\,n-3}{2}.\ \text{cos.}\,\overline{n-4}.\ A$, &c. $= 2^{n-1}\,x^n - n.$ $\text{cos.}\,\overline{n-2}\,.\,A - \frac{n\,.\,n-1}{2}.\ \text{cos.}\,\overline{n-4}\,.\,A -$ &c. therefore

$$x^n = \frac{1}{2^{n-1}} \times \overline{\text{cos.}\,nA + n.\ \text{cos.}\,\overline{n-2}.A + \frac{n.n-1}{2} \times \text{cos.}\,\overline{n-4}.\,A\ \&c.}$$

The coefficients are the same as the unciæ in the power n of a binomial, only in the last term we must take but half the correspondent uncia

uncia, the reason of which is, that the last of the arcs, nA, $\overline{n-2}\,.\,A$, $\overline{n-4}\,.\,A$, &c. will be o, the cosine of which $=1$, but by the theorem in art. 237 it is $\frac{1}{2}$, because we made $z=Q^n$, which may be done when n is greater than o, but when $n=0$, $\frac{r^2}{Q}=\frac{1}{2x}+\frac{1}{8x^2}$ &c. and $\frac{r^2n}{Q^n}=1$ which added to $Q^n=1$ gives $2=2z$ and $z=1$, as it should be.

240. Suppose the whole circumference to be divided into as many equal parts Ab, bc, cd, &c. (fig. 66.) as there are units in n, and call the whole circumference C, the arc $Ab=\frac{C}{n}$,A; then $Ab=\frac{C}{n}$, $Ac=\frac{2C}{n}$, $Ad=\frac{3C}{n}$, &c. call the cosines of the arcs Ab, Ac, Ad, &c. $\pm a$, $\pm b$, $\pm c$ &c. each of which is to be affirmative or negative according as it is on the same side of A as C or not. Now since the arcs C, 2C, 3C, &c. have all the same cosine AC or r, the relation of r to a will be found by exterminating Q out of the two equations $Q^{2n}-2r^nQ^n+r^{2n}=0$, and $Q^2-2aQ+r^2=0$; and the relation of r to b will be found by the two equations $Q^{2n}-2r^n Q^n+r^2=0$; and $Q^2-2bQ+r^2=0$ and so the relation of r to c is found by the equations $Q^{2n}-2r^n Q^n+r^{2n}=0$ and $Q^2-2cQ+r^{2n}=0$. Therefore the quantities $Q^2-2aQ+r^2$, $Q^2-2bQ+r^2$, $Q^2-2cQ+r^2$, &c. are divisors of the quantity $Q^{2n}-2r^nQ^n+r^{2n}$. And the quantity $Q^{2n}-2r^nQ^n+r^{2n}$ is equal to the product of all the quantities $Q^2-2aQ+r^2$, $Q^2-2bQ+r^2$, $Q^2-2cQ+r^2$, &c.

241.

241. On CA take $CQ = Q$, then if $b\beta$ is the sine of Ab, $C\beta = a$, and $Q\beta = a - Q$, and $Q\beta^2 = a^2 - 2aQ + Q^2$, therefore $Q^2 - 2aQ + r^2 = \beta Q^2 + r^2 - a^2 = \beta Q^2 + \beta b^2 = Qb^2$. In the same manner if $c\varkappa$ is the sine of Ac, then $Q\varkappa = Q + C\varkappa = Q - b$ and $Q^2 - 2bQ + b^2 = \varkappa Q^2$ and $Q^2 - 2bQ + r^2 = \varkappa Q^2 + r^2 - b^2 = cQ^2$, and so $Q^2 - 2cQ + r^2 = Qd^2$ &c. therefore we have $Qb^2 \times Qc^2 \times Qd^2 \times$ &c. $= Q^{2n} - 2r^n Q^n + r^{2n}$, and extracting the square root $Qb \times Qc \times Qd \times$ &c. $= Q^n - r^n$ or $r^n - Q^n$ according as Q is greater or less than r. That is $Qb \times Qc \times Qd \times$ &c. $= CQ^n - AC^n$ or $AC^n - CQ^n$.

242. If we suppose all the arcs Ab, bc, cd, &c. to be bisected in b, c, d, &c. then n will now be be $2n$, and by the last article $Qb \times Qb \times Qc \times Qc \times Qd \times Qd \times$ &c. $= r^{2n} - Q^{2n}$. But $Qb \times Qc \times Qd \times$ &c. $= r^n - Q^n$. Therefore $Qb \times Qc \times$

$$Qd \times \text{\&c.} = \frac{r^{2n} - Q^{2n}}{r^n - Q^n} = r^n + Q^n = AC^n + CQ^n.$$

This is the elegant theorem invented by Mr. COTES*.

243. If again the arcs Ab, bb, bc, cc, cd, dd, &c. are bisected in the points β, B, $\varkappa$, C, δ, D, &c. (fig. 67) then n will $= 4n$ and $Q\beta \times Qb \times QB \times Qb \times Q\varkappa \times Qc \times QC \times Qc \times$ &c. $= r^{4n} - Q^{4n}$. But $Qb \times Qb \times Qc \times Qc \times$ &c. $= r^{2n} - Q^{2n}$ therefore $Q\beta \times QB \times Q\varkappa \times QC \times Q\delta \times QD \times$ &c. $=$

$$\frac{r^{4n} - Q^{4n}}{r^{2n} - Q^{2n}} = r^{2n} + Q^{2n}.$$

And since $Q\beta = QG$, $Q\varkappa = QF$, &c. therefore $QB^2 \times QC^2 \times QD^2 \times$ &c. $= r^{2n} + Q^{2n}$.

* Harm. Mensur. p. 114.

Scholium.

Scholium.

244. Hence we may find the divisors of a trinomial as $Q^{2n} - 2r^{n-1} zQ^n + r^{2n}$. In a circle whose radius is r, take an arc ϕ whose cosine is z and an arc $A = \frac{\phi}{n}$ then $x = \text{cos.} \frac{\phi}{n}$ and the divisor $Q^2 - 2xQ + r^2$ will become $Q^2 - 2Q \times \text{cos.} \frac{\phi}{n} + r^2$ or if we take the cosines in a circle whose radius is 1, then the trinomial may be reduced to $r^{2n} \times \overline{\frac{Q^{2n}}{r^{2n}} - \frac{2zQ^n}{r^{n+1}} + 1}$, and take an arc ϕ whose cosine $= \frac{z}{r}$ and the divisor will be $r^2 \times \overline{\frac{Q^2}{r^2} - \frac{2Q}{r} \times \text{cos.} \frac{\phi}{n} + 1} = Q^2 - 2rQ \times \text{cos.} \frac{\phi}{n} + r^2$. Let C be the whole circumference then the arcs ϕ, $C + \phi$, $2C + \phi$, &c. will have the same cosine as ϕ and if k be any whole number not greater than n, the arc $kC \pm \phi$ will have the same cosine as ϕ, and the general expression for the divisor is $Q^2 - 2rQ \times \text{cos.} \frac{kC \pm \phi}{n} + r^2$, and all the divisors will be found by substituting for k all the whole numbers not greater than n.

245. If $z = r$ then the trinomial will be $Q^{2n} - 2r^n Q^n + r^{2n}$ and extracting the square root, the binomial $Q^n - r^n$ will have the same divisors as the trinomial $Q^{2n} - 2r^n Q^n + r^{2n}$. And as $z = r$, the

the arc $\phi = 0$ and the general expression for the divisor will be $Q^2 - 2rQ \times \text{cos.}\, \frac{kC}{n} + r^2$.

246. If $z = 0$; then $\phi = \frac{1}{4}C$ and the trinomial will become the binomial $Q^{2n} + r^{2n}$ whose divisor will be $Q^2 - 2rQ \times \text{cos.}\, \frac{kC \pm \frac{1}{4}C}{n} + r^2$ or $Q^2 - 2rQ \times \text{cos.}\, \frac{4k \pm 1}{4n} \times C + r^2$.

247. Let the modular ratio (Prop. XIV. Cor. 2.) be that of e to 1, and let q be an infinite number, then by art. 91. $e^x - e^{-x} = \overline{1 + \frac{x}{q}}\big|^q - \overline{1 - \frac{x}{q}}\big|^q$ which compared with the above binomial $Q^n - r^n$ gives $Q = 1 + \frac{x}{q}$, $r = 1 - \frac{x}{q}$, $n = q$ and its divisor is $\overline{1 + \frac{x}{q}}\big|^2 - 2 \times \overline{1 - \frac{x^2}{q^2}} \times \text{cos.}\, \frac{kC}{q} + \overline{1 - \frac{x}{q}}\big|^2 = 2 \times \overline{1 + \frac{x^2}{q^2}} - 2 \times \overline{1 - \frac{x^2}{q^2}} \times \text{cos.}\, \frac{kC}{q}$. But by art. 233. Cor. 2. we have $\text{cos.}\, \frac{kC}{q} = 1 - \frac{k^2C^2}{2q^2}$ the other terms vanishing because q is infinite, therefore the divisor will be $2 + \frac{2x^2}{q^2} - 2 + \frac{2x^2}{q^2} + \frac{k^2C^2}{q^2} - \frac{k^2x^2C^2}{q^4} = \frac{4x^2}{q^2} + \frac{k^2C^2}{q^2} - \frac{k^2x^2C^2}{q^4}$ and multiplying by $\frac{q^2}{k^2C^2}$ the divisor will be $1 + \frac{4x^2}{k^2C^2} - \frac{x^2}{q^2} = 1 + \frac{4x^2}{k^2C^2}$ since

$\frac{x^2}{q^2}$ is infinitely small, and by substituting 1, 2, 3, &c. for k we shall have all the divisors, but it appears by art. 91. that the quantity $\overline{e^x - e^{-x}}$ has a divisor, $2x$, therefore $\frac{e^x - e^{-x}}{2} = x \times \overline{1 + \frac{4x^2}{C^2}} \times \overline{1 + \frac{4x^2}{4C^2}} \times \overline{1 + \frac{4x^2}{9C^2}} \times$ &c. $= x \times \overline{1 + \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5}}$, &c.

248. In the same manner $e^x + e^{-x} = \overline{1 + \frac{x}{q}}|^q + \overline{1 - \frac{x}{q}}|^q$, this compared with the binomial $Q^{2n} + r^{2n}$, gives $Q = 1 + \frac{x}{q}$, $r = 1 - \frac{x}{q}$, $n = \frac{1}{2}q$ and the divisor is $\overline{1 + \frac{x}{q}}|^2 - 2 \times \overline{1 - \frac{x^2}{q^2}} \times \text{cos.}\, \frac{4k \pm 1}{2q} \times C + \overline{1 - \frac{x}{q}}|^2 = 2 + \frac{2x^2}{q^2} - 2 \times \overline{1 - \frac{x^2}{q^2}} \times \text{cos.}\, \frac{4k \pm 1}{2q} \times C$. But cosine $\frac{4k \pm 1}{2q} \times C = 1 - \frac{\overline{4k \pm 1}|^2 \times C^2}{8q^2}$; therefore the divisor will be $\frac{4x^2}{q^2} + \frac{\overline{4k \pm 1}|^2 \times C^2}{4q^2} - \frac{\overline{4k \pm 1}|^2 \times C^2 x^2}{4q^4} = \frac{4x^2}{q^2} + \frac{\overline{4k \pm 1}|^2 \times C^2}{4q^2}$, multiply by $\frac{4q^2}{\overline{4k \pm 1}|^2 \times C^2}$ then the divisor will be $1 +$

$16x^2$

$\frac{16x^2}{\overline{4k\pm1}|^2 \times C^2}$ and substituting all the odd numbers for $4k\pm1$, and as $e^x + e^{-x}$ has also the divisor 2, we shall have $\frac{e^x + e^{-x}}{2} = 1 + \frac{16x^2}{C^2} \times \overline{1 + \frac{16x^2}{9C^2}} \times \overline{1 + \frac{16x^2}{25C^2}} \times$ &c. $= 1 + \frac{x^2}{2} + \frac{x^4}{2.3.4}$, &c.

249. By art. 233. Cor. 1. the sine of the arc x in a circle whose radius is 1, is equal to $x - \frac{x^3}{6} + \frac{x^5}{120}$, &c. $=$ (by art. 91.) $\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$ by changing the sign of x^2 in art. 247, we shall have, sin. $x = x \times \overline{1 - \frac{4x^2}{C^2}} \times \overline{1 - \frac{4x^2}{4C^2}} \times \overline{1 - \frac{4x^2}{9C^2}} \times$ &c.

The cosine of the arc x is equal $1 - \frac{x^2}{2} + \frac{x^4}{24}$ &c. $= \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} = \overline{1 - \frac{16x^2}{C^2}} \times \overline{1 - \frac{16x^2}{9C^2}} \times \overline{1 - \frac{16x^2}{25C^2}} \times$ &c.

250. If we multiply the divisors in the expression for the sine we shall have $1 - \frac{1}{6}x^2$, &c. $= 1 - \frac{4x^2}{C^2} \times \overline{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +}$ &c. therefore $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ &c. $= \frac{C^2}{24}$.

In the same manner in the expression for the cosine we have $1 - \frac{x^2}{2}$ &c. $= 1 - \frac{16x^2}{C^2} \times$

 $1+$

$1 + \frac{1}{9} + \frac{1}{25}$ &c. and $1 + \frac{1}{9} + \frac{1}{25}$ &c. $= \frac{C^2}{32}$. This series subtracted from the last series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ &c. $= \frac{C^2}{24}$ leaves $\frac{1}{4} + \frac{1}{16} + \frac{1}{36}$ &c. $= \frac{C^2}{96}$ and this subtracted from $1 + \frac{1}{9} + \frac{1}{25}$ &c. $= \frac{C^2}{32}$ leaves $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16}$ &c. $= \frac{C^2}{48}$.

251. Let the arc $x = nC$, then the sine of $x = nC \times \overline{1 - 4n^2} \times \overline{1 - \frac{4n^2}{4}} \times \overline{1 - \frac{4n^2}{9}} \times$ &c. and

$$C = \frac{\text{Sin. } nC}{n \times \overline{1 - 4n^2} \times \overline{1 - \frac{4n^2}{4}} \times \overline{1 - \frac{4n^2}{9}} \times \text{\&c.}}$$

$$= \text{sin. } nC \times \frac{1}{n} \times \frac{1}{1-2n} \times \frac{1}{1+2n} \times \frac{2}{2-2n} \times \frac{2}{2+2n} \times \frac{3}{3-2n} \times \frac{3}{3+2n} \text{ \&c.}$$

If $n = \frac{1}{4}$ then sin. $nC =$ sin. $90° = 1$ and $C =$

$$\frac{1}{\frac{1}{4} \times \overline{1 - \frac{1}{4}} \times \overline{1 - \frac{1}{16}} \times \overline{1 - \frac{1}{36}}} \text{ \&c.} = 1 \times 4 \times \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \text{ \&c.} = 8 \times \frac{8}{9} \times \frac{24}{25} \text{ \&c.} = 8 \times \overline{1 - \frac{1}{9}} \times \overline{1 - \frac{1}{25}} \times \text{\&c.}$$

If $n = \frac{1}{12}$ then sin. $nC =$ sin. $30° = \frac{1}{2}$ and $C =$

6

$$\frac{6}{\overline{1-\frac{1}{6^2}}\times\overline{1-\frac{1}{12^2}}\times\overline{1-\frac{1}{18^2}}\times \&c.} = 6\times\frac{6}{5}\times\frac{6}{7}\times\frac{12}{11}\times\frac{12}{13}\times \&c.$$

252. Since cos. $nC =$ sin. $\frac{1}{4}\overline{C-nC} =$ sin. $\frac{1-4n}{4}.C.$ and as sine $nC = nC\times\overline{1-2n}\times\overline{1+2n}\times\frac{2-2n}{2}\times\frac{2+2n}{2}\times\frac{3-2n}{3}\times\frac{3+2n}{3}$, &c. If we put $\frac{1}{4}-n$ for n and $\frac{1}{2}-2n$ for $2n$ we shall have cos. $nC = \frac{1-4n}{4}\times C\times\frac{1+4n}{2}\times\frac{3-4n}{2}\times\frac{3+4n}{4}\times\frac{5-4n}{4}\times$ &c. And $C =$ cos. $nC\times\frac{4}{1-4n}\times\frac{2}{1+4n}\times\frac{2}{3-4n}\times\frac{4}{3+4n}\times\frac{4}{5-4n}\times$ &c.

Also tang. $nC = \frac{\text{sin. } nC}{\text{cos. } nC} = \frac{4n}{1-4n}\times\frac{2-4n}{1+4n}\times\frac{2+4n}{3-4n}\times$ &c.

253. If the arc Ab (fig. 66.) is to the arc AP as 1 to n, and the tangent of A$b=t$, the tangent of AP$=v$, and let radius $=r$, then by art. 234, A$b=$ $r\sqrt{-1}\,\Big|\frac{r\sqrt{-1}+t}{\sqrt{r^2+t^2}} = r\sqrt{-1}\,\Big|\frac{r-t\sqrt{-1}}{\sqrt{-r^2-t^2}} = \frac{1}{2}r\sqrt{-1}\,\Big|\frac{\overline{r-t\sqrt{-1}}^2}{-r^2-t^2} = \frac{1}{2}r\sqrt{-1}\,\Big|\frac{r-t\sqrt{-1}}{r+t\sqrt{-1}}$ and

$AP = n \times Ab = \frac{1}{2}v\sqrt{-1}\,\Big|\frac{r - v\sqrt{-1}}{r + v\sqrt{-1}}$. Put $\frac{r - t\sqrt{-1}}{r + t\sqrt{-1}} = \frac{N}{P}$ then $n\Big|\frac{N}{P} = 1\,\Big|\frac{r - v\sqrt{-1}}{r + v\sqrt{-1}}$, therefore $\frac{P^n}{N^n} = \frac{r + v\sqrt{-1}}{r - v\sqrt{-1}}$ and $rP^n - vP^n\sqrt{-1} = rN^n + vN^n\sqrt{-1}$ whence $\overline{P^n - N^n} \times r = \overline{P^n + N^n} \times v\sqrt{-1}$ and $v = \frac{P^n - N^n}{P^n + N^n} \times \frac{r}{\sqrt{-1}}$.

But by Lemma 2. $P^n = r^n + nr^{n-1}t\sqrt{-1} - \frac{n.n-1}{2}r^{n-2}t^2 - \frac{n.n-1.n-2}{2.3}r^{n-3}t^3\sqrt{-1} +$ &c. and $N^n = r^n - nr^{n-1}t\sqrt{-1} - \frac{n.n-1}{2}r^{n-2}t^2 + \frac{n.n-1.n-2}{2.3}r^{n-3}t^3\sqrt{-1}$, &c. therefore $v =$

$$\frac{nr^n t - \frac{n.n-1.n-2}{2.3} \times r^{n-2}t^3 + \text{\&c.}}{r^n - \frac{n.n-1}{2}.r^{n-2}t^2 + \frac{n.n-1.n-2.n-3}{2.3.4}.r^{n-4}t^4}$$

&c. $=$ (by division) $nt + \frac{n+1.n.n-1}{3}.\frac{t^3}{r^2} + \frac{n+1.n.n-1.2n^2-3}{3.5}.\frac{t^5}{r^4}$, &c.

254. Let the arc be A and let y and x be its sine and cosine, then $t = \frac{ry}{x}$ which being put for t we have v equal to

nr

$$\frac{\frac{nr^{n+1}y}{x} - \frac{n.n-1.n-2}{2.3} \times \frac{r^{n+1}y^3}{x^3} + \&c.}{r^n - \frac{n.n-1}{2}.\frac{r^ny^2}{x^2} + \frac{n.n-1.n-2.n-3}{2.3.4}.\frac{r^ny^4}{x^4}\ \&c.}$$

$$= r \times \frac{x^{n-1}y - \frac{n.n-1.n-2}{2.3}.x^{n-3}y^3 + \&c.}{x^n - \frac{n.n-1}{2}.x^{n-2}y^2 + \&c.} = =$$

tang. nA. But by art. 236. $r^{n-1} \times$ cos. nA $=$ $x^n - \frac{n.n-1}{2}.x^{n-2}y^2 + \&c.$ therefore sin. nA $=$ $\frac{\text{tang. } nA \times \text{cos. } nA}{r} = \frac{x^{n-1}y}{r^{n-1}} - \frac{n.n-1.n-2}{2.3} \times \frac{x^{n-3}y^3}{r^{n-1}} + \&c.$

255. Since $t = \frac{ry}{x}$ we have by art. 253. tang. $nA = \frac{\overline{x+y\sqrt{-1}}|^n - \overline{x-y\sqrt{-1}}|^n}{\overline{x+y\sqrt{-1}}|^n + \overline{x-y\sqrt{-1}}|^n} \times \frac{r}{\sqrt{-1}}$ and by art. 236. cos. $nA =$ $\frac{\overline{x+y\sqrt{-1}}|^n + \overline{x-y\sqrt{-1}}|^n}{2r^{n-1}}$ therefore sin. $nA = \frac{\text{tang. } nA \times \text{cos. } nA}{r} =$ $\frac{\overline{x+y\sqrt{-1}}|^n - \overline{x-y\sqrt{-1}}|^n}{2r^{n-1}\sqrt{-1}}$ = (if we put $Q = x+y\sqrt{-1}$ as in art. 235.) $\frac{Q^n}{2r^{n-1}\sqrt{-1}}$ —

$$\frac{r^{n+1}}{2Q^r\sqrt{-1}} = \frac{y}{2r^{n-1}} \times \overline{\frac{Q^n}{y\sqrt{-1}} - \frac{r^{2n}}{yQ^n\sqrt{-1}}}.$$

$= \frac{y}{2r^{n-1}} \times Q^n \times \overline{x^2 - r^2}|^{-\frac{1}{2}}$. But by art. 237.

$Q^n = 2^n . x^n - n . 2^{n-2} x^{n-2} r^2 + \frac{n . \overline{n-3}}{2} . 2^{n-4} x^{n-4} r^4 - \frac{n . \overline{n-4} . \overline{n-5}}{2 . 3} . 2^{n-6} x^{n-6} r^6$ &c.

which multiplied by $\overline{x^2 - r^2}|^{-\frac{1}{2}} = \frac{1}{x} + \frac{r^2}{2x^3} + \frac{3r^4}{8x^5} + \frac{5r^6}{16x^7} +$ &c. gives $\frac{Q^n}{y\sqrt{-1}} = 2^n x^{n-1} - \overline{n-2} . 2^{n-2} x^{n-3} r^2 + \frac{\overline{n-3} . \overline{n-4}}{2} . 2^{n-4} x^{n-5} r^4 - \frac{\overline{n-4} . \overline{n-5} . \overline{n-6}}{2.3} . 2^{n-6} x^{n-7} r^6$, &c. and fin.

$nA = \frac{y}{2r^{n-1}} \times \frac{Q^n}{y\sqrt{-1}} = y$ into $\frac{2^{n-1} x^{n-1}}{r^{n-1}} - \frac{\overline{n-2} . 2^{n-3} x^{n-3}}{r^{n-3}} + \frac{\overline{n-3} . \overline{n-4}}{2} \times \frac{2^{n-5} x^{n-5}}{r^{n-5}} - \frac{\overline{n-4} . \overline{n-5} . \overline{n-6}}{2.3} \times \frac{2^{n-7} x^{n-7}}{r^{n-7}}$ &c.

If $r = 1$. fin. $nA = y \times \overline{2^{n-1} x^{n-1} - \overline{n-2} . 2^{n-3} x^{n-3} + \frac{\overline{n-3}.\overline{n-4}}{2} \times 2^{n-5} x^{n-5}}$ &c.

Example

Example IV. *Fig.* 62.

256. *To find the length of an ellipse.*

Call the greater semiaxis AC, a, the lesser CD, b, CO, x, OP, y, then $y^2 = \frac{bb}{aa} \times \overline{a^2 - x^2}$ and $y\dot{y} = \frac{-b^2x\dot{x}}{a^2}$, whence $\dot{y}^2 = \frac{b^4x^2\dot{x}^2}{a^4y^2} = \frac{b^2x^2\dot{x}^2}{a^2 \times \overline{a^2 - x^2}}$ and $\dot{x}^2 + \dot{y}^2 = \dot{x}^2 \times \frac{a^4 - a^2x^2 + b^2x^2}{a^2 \times \overline{a^2 - x^2}}$, put $c^2 = \frac{a^2 - b^2}{a^2}$ then $\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}\sqrt{a^2 - c^2x^2}}{\sqrt{a^2 - x^2}}$, the fluent of which by art. 200. is equal to aA $- \frac{c^2\mathrm{B}}{2a} - \frac{c^4\mathrm{C}}{8a^3}$ — &c. where A is the fluent of $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$ or if A is equal to the circular arc AM then $\dot{\mathrm{A}} = \frac{a\dot{x}}{\sqrt{a^2 - x^2}}$ and the arc AP $= \mathrm{A} - \frac{c^2\mathrm{B}}{2a^2} - \frac{c^4\mathrm{C}}{8a^4}$ &c. and when $x = a$, then the length of the quadrant of the ellipse is equal to A into $1 - \frac{c^2}{4} - \frac{3c^4}{64} - \frac{5c^6}{256}$ &c. where A is equal to the quadrant AMH.

Example

Example V. *Fig.* 7.

257. *To find the length of an hyperbola.*

Call the semi-transverse axis SA, a, the semi-conjugate, SG, b, SO, x, OP, y, then $y^2 = \frac{b^2}{a^2} \times \overline{x^2 - a^2}$ and $\dot{y}^2 = \frac{b^4 x^2 \dot{x}^2}{a^4 y^2} = \frac{b^2 x^2 \dot{x}^2}{a^2 \times \overline{x^2 - a^2}}$

therefore $\dot{x}^2 + \dot{y}^2 = \dot{x}^2 \times \frac{-a^4 + a^2 x^2 + b^2 x^2}{a^2 \times \overline{x^2 - a^2}}$,

Put $\frac{a^2 + b^2}{a^2} = c^2$ then $\sqrt{\dot{x}^2 + \dot{y}^2} =$

$\frac{\dot{x}\sqrt{c^2 x^2 - a^2}}{\sqrt{x^2 - a^2}} = \frac{\dot{x}\sqrt{a^2 - c^2 x^2}}{\sqrt{a^2 - x^2}} =$

$\dot{x}\sqrt{1 - \frac{c^2 - 1}{a^2 - x^2} \times x^2} = \dot{x}\sqrt{1 - \frac{b^2 x^2}{a^4 - a^2 x^2}} =$

$\dot{x}\sqrt{1 - \frac{b^2 x^2}{a^4} - \frac{b^2 x^4}{a^6}}$, &c. $= \dot{x} \times 1 - \frac{b^2 x^2}{2a^4} -$

$\frac{b^4 x^4 + 4a^2 b^2 x^4}{8a^8}$ &c. whose fluent is $x - \frac{b^2 x^3}{6a^4} -$

$\frac{b^4 x^5 + 4a^2 b^2 x^5}{40a^8}$ &c. $+$ C which is to vanish when $x = a$, therefore $-\text{C} = a - \frac{b^2}{6a} - \frac{b^4 + 4a^2 b^2}{40a^3}$

&c. and the arc AP is equal to the difference between the two serieses $x - \frac{b^2 x^3}{6a^4} - \frac{b^4 x^5 + 4a^2 b^2 x^5}{40a^8}$

&c. and $a - \frac{b^2}{6a} - \frac{b^4 + 4a^2 b^2}{40a^3}$ &c. Q. E. I.

Example

Example VI. *Fig.* 72.

258. *To find the length of the* Cissoid *of* Diocles.

Call AB, a, AO, x, OP, y, then by art. 111. we have $\dot{y} = \frac{\dot{x}y}{2x^3} \times \overline{3x^2 + y^2} = \frac{\dot{x}y \times \overline{3a - 2x}}{2x \times \overline{a - x}}$ and $\dot{y}^2 = \frac{\dot{x}^2 y^2 \times \overline{9a^2 - 12ax + 4x^2}}{4x^2 \times \overline{a-x}|^2}$. But $\frac{y^2}{x^2} = \frac{x}{a-x}$ therefore $\dot{y}^2 = \dot{x}^2 \times \frac{9a^2x - 12ax^2 + 4x^3}{4 \times \overline{a-x}|^3}$ and $\dot{x}^2 + \dot{y}^2 = \dot{x}^2 \times$
$\frac{9a^2x - 12ax^2 + 4x^3 + 4a^3 - 12a^2x^2 + 12ax^2 - 4x^3}{4 \times \overline{a - x}|^3} =$
$\dot{x}^2 \times \frac{4a^3 - 3a^2x}{4 \times \overline{a-x}|^3}$, put $a - x$ or BO $= z$, then $\dot{x} = -\dot{z}$ and $4a - 3x = a + 3z$ therefore $\dot{x}^2 + \dot{y}^2 = \frac{a^2\dot{z}^2 \times \overline{a + 3z}}{4z^3}$ and $\sqrt{\dot{x}^2 + \dot{y}^2} = -\frac{1}{2} a\dot{z}z^{-\frac{3}{2}} \times \sqrt{a + 3z}$ which may be compared with the fluxion in art. 94. Exam. 8. where $d = \frac{-a}{2}$, $\eta = 1$, $e = a$, $f = 3$, $R = \sqrt{3}$, $T = N = \sqrt{\frac{a + 3z}{z}}$, $S = \sqrt{\frac{a}{z}}$, and the fluent is $\frac{-2d}{\eta} N +$

$$\frac{2d}{\eta} R \left| \frac{R + T}{S} \right. = aN - a\sqrt{3} \left| \frac{\sqrt{3} + N}{\sqrt{\frac{a}{z}}} \right. + C$$

which

which is to vanish when $z = a$, and $N = 2$, therefore $C = -2a + a\sqrt{3}\,\Big|\dfrac{\sqrt{3}+2}{1}$ and the whole fluent $= AP = aN - 2a -$

$$a\sqrt{3}\,\Bigg|\frac{\sqrt{3}+N}{\sqrt{\frac{3a}{z}}+2\sqrt{\frac{a}{z}}},\ = 2\times\overline{\tfrac{1}{2}aN - a} +$$

$$a\sqrt{3}\,\Bigg|\frac{a+2a\sqrt{\tfrac{1}{3}}}{\sqrt{az}+N\sqrt{\tfrac{1}{3}az}}.$$

Draw AE through the middle of OP meeting the asymptote in E then $BE^2 = \dfrac{OP^2 \times AB^2}{4AO^2} = \dfrac{a^2x^3}{4x^2z} = \dfrac{a^2x}{4z}$ and $AE^2 = a^2 + \dfrac{a^2x}{4z} = a^2 \times \dfrac{4z+x}{4z} = a^2 \times \dfrac{a+3z}{4z}$, therefore $AE = \tfrac{1}{2}aN$. Let the angle BAF be $\tfrac{1}{3}$ of a right angle, then $BF = a\sqrt{\tfrac{1}{3}}$, $3BF = a\sqrt{3}$, and $AF = 2a\sqrt{\tfrac{1}{3}}$. Let BD be a mean proportional between AB and BO then $BD = \sqrt{az}$, and $DF^2 = az + \dfrac{a^2}{3} = \dfrac{az}{3} \times \overline{3 + \dfrac{a}{z}}$, and $DF = N\sqrt{\tfrac{1}{3}az}$ therefore the arc

$$AP = 2 \times \overline{AE - AB} + 3BF\,\Bigg|\frac{AB + AF}{DB + DF}\ *.$$

* Harm. Mensur. p. 114.

Example

Example VII. *Fig.* 8.

259. *To find the length of the* Conchoid *of* Nicomedes.

Retaining the same symbols as in art. 112. we have $\dot{y} = \frac{\dot{x}}{x^2 v} \times \overline{a^2 b + x^3}$ and $\dot{y}^2 =$
$\frac{\dot{x}^2 \times \overline{a^2 b + x^3}^2}{x^4 v^2} = \frac{\dot{x}^2 \times \overline{a^4 b^2 + 2a^2 b x^3 + x^6}}{x^4 \times \overline{a^2 - x^2}}$
therefore $\dot{x}^2 + \dot{y}^2 = \dot{x}^2 \times$
$\frac{a^4 b^2 + 2a^2 b x^3 + a^2 x^4}{x^4 \times \overline{a^2 - x^2}}$, and $\sqrt{\dot{x}^2 + \dot{y}^2} =$
$\frac{a\dot{x}}{x^2}\sqrt{\frac{a^4 b^2 + 2bx^3 + x^4}{a^2 - x^2}} =$
$\frac{a\dot{x}}{x^2}\sqrt{b^2 + \frac{b^2 x^2}{a^2} + \frac{2bx^3}{a^2} + \&c.} = -\frac{a\dot{x}}{x^2} \times$
$\overline{b + \frac{bx^2}{2a^2} + \frac{x^3}{a^2} \&c.} = -\dot{x} \times$
$\overline{\frac{ab}{x^2} + \frac{b}{2a} + \frac{x}{a} + \&c.}$ whose fluent is $\frac{ab}{x} -$
$\frac{bx}{2a} - \frac{x^2}{2a}$ &c. $+ C$ which is to vanish when $x = a$, therefore $C = -b + \frac{b}{2} + \frac{a}{2}$ &c. $=$
$-\frac{b}{2} + \frac{a}{2}$ &c. and $AP = \frac{ab}{x} - \frac{bx}{2a} - \frac{x^2}{2a}$
&c. $- \frac{b}{2} + \frac{a}{2}$ &c. Q. E. I.

Example

Example VIII. *Fig.* 73.

260. *To find the length of the logarithmic curve.*

Let AV be the first ordinate, P any point in the curve, VF, PT, two tangents in the points V and P, call the subtangent AF, a, AO, x, OP, y, then by art. 115, $\dot{x}y = a\dot{y}$, and $\dot{x}^2 = \frac{a^2\dot{y}^2}{y^2}$ therefore $\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{y}\sqrt{1 + \frac{a^2}{y^2}}$, which may be compared with the fluxion in art. 94. Exam. 8. where $d = 1$, $n = -2$, $e = 1$, $f = a^2$, $R = a$, $T = N = y\sqrt{1 + \frac{a^2}{y^2}} = \sqrt{a^2 + y^2} =$ PT, $S = y$, and the fluent $\frac{-2}{n} dN + \frac{2}{n} dR \left| \frac{R+T}{S} \right. = PT - a \left| \frac{a+PT}{y} \right. + C$, which is to vanish when $x = 0$, or $y = AV$, therefore $C = -VF + a \left| \frac{a+VF}{AV} \right.$ and the whole fluent or the arc $VP = PT - VF - AF \left| \frac{AF + PT}{OP} \right. + AF \left| \frac{AF + VF}{AV} \right. = PT - VF - AF \left| \frac{OP}{PT - AF} \right. + AF \left| \frac{AV}{VF - AF} \right.$.

Let AL be $=$ VF $-$ AF, and Ol $=$ PT$-$OT and draw LM, lm parallel to the asymptote meeting

ing the curve in M and m, then LM $=$ AF $\left|\frac{AV}{AL}\right.$, and $lm =$ AF $\left|\frac{OP}{Ol}\right.$ and the arc VP $=$ PT $-$ VF $+$ LM $-$ lm *.

Example IX. *Fig.* 17.

261. *To find the length of a Cycloid.*

Retaining the same symbols as in art. 117. we have $\dot{y} = \frac{\dot{x} \times \overline{b+x}}{\sqrt{a^2-x^2}}$, and $\dot{y}^2 = \dot{x}^2 \times \frac{b^2+2bx+x^2}{a^2-x^2}$ whence $\dot{x}^2+\dot{y}^2 = \dot{x}^2 \times \frac{a^2+b^2+2bx}{a^2-x^2}$, put $a^2+b^2=c^2$ then $\sqrt{\dot{x}^2+\dot{y}^2} = \dot{x}\sqrt{\frac{c^2+2bx}{a^2-x^2}} = \dot{x}\sqrt{\frac{c^2}{a^2}+\frac{2bx}{a^2}+\frac{c^2x^2}{a^4}\ \&c.} = \dot{x} \times \frac{c}{a}+\frac{bx}{ac}+\frac{c^4-a^2b^2}{2a^3c^3} \times x^2$ &c. The fluent of which is $\frac{cx}{a}+\frac{bx^2}{2ac}+\frac{c^4-a^2b^2}{6a^3c^3} \times x^3$, &c. $+$C which is to vanish when $x=a$, therefore C $= -c - \frac{ab}{2c} - \frac{c^4-a^2b^2}{6c^3}$ &c. and the arc VP $= \frac{cx}{a} + \frac{bx^2}{2ac} + \frac{c^4-a^2b^2}{6a^3c^3} \times x^3$ &c. $- c - \frac{ab}{2c} - \frac{c^4-a^2b^2}{6c^3}$ &c. Q. E. I.

* Cotes Harm. Mensur. p. 23.

262. In the common cycloid $a=b$ and $c^2=2a^2$ therefore $\sqrt{\dot{x}^2+\dot{y}^2} = \dot{x}\sqrt{\frac{2a^2+2ax}{a^2-x^2}}, =$ $\dot{x}\sqrt{\frac{2a}{a-x}}$ the fluent of which is $2\sqrt{2a\times\overline{a-x}} =$ $2\sqrt{AL\times OL} = 2Lm$, therefore the arc VP is equal to twice the chord Lm as we proved before in art. 173.

Example X. *Fig.* 25.

263. *To find the length of the Quadratrix.*

Supposing the same things as in art. 218 we have $\dot{x} = -\dot{y}\times\overline{\frac{2y}{3r}+\frac{4y^3}{45r^3}+\frac{4y^5}{315r^5}}$ &c. and $\dot{x}^2=\dot{y}^2\times\overline{\frac{4y^2}{9r^2}+\frac{16y^4}{135r^4}}$ &c. therefore $\dot{x}^2+\dot{y}^2=$ $\dot{y}^2\times\overline{1+\frac{4y^2}{9r^2}+\frac{16y^4}{135r^4}}$ &c. and $\sqrt{\dot{x}^2+\dot{y}^2}=\dot{y}\times$ $\overline{1+\frac{2y^2}{9r^2}+\frac{14y^4}{405r^4}}$ &c. the fluent of which is $y+$ $\frac{2y^3}{27r^2}+\frac{14y^5}{2025r^4}$ &c. $=$ AP. Q. E. I.

PROP.

PROP. XLIV.

264. *Let* SWP (*fig.* 21.) *be a spiral, and retaining the same symbols as in Prop.* XXII. *I say the fluxion of the arc* SWP *is equal to* $\sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{r^2}}$.

For since the triangles Pnp, PST are similar, SP : PT :: nP : Pp :: $\dot{y}$: $\dot{t}$ and $\dot{t} = \frac{\dot{y}}{y} \times$ PT, but PT $= \sqrt{y^2 + \frac{y^4\dot{x}^2}{r^2\dot{y}^2}} = \frac{y}{\dot{y}} \times \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{r^2}}$ therefore $\dot{t} = \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{r^2}}$. Q. E. D.

265. Let $x : r :: y^m : a^m$, then $\dot{x} = \frac{mr\dot{y}y^{m-1}}{a^m}$, and $\frac{y\dot{x}}{r} = \frac{m\dot{y}y^m}{a^m}$ therefore $\sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{r^2}} =$ $\dot{y}\sqrt{1 + \frac{m^2y^{2m}}{a^{2m}}}$ the fluent of which is equal to the arc SWP.

266. If $m = 1$, then the fluxion of SWP is $\dot{y}\sqrt{1 + \frac{y^2}{a^2}}$ which may be compared with the fluxion in art. 94. exam. 6. where $d = 1$, $n = 2$, $e = 1$, $f = \frac{1}{a^2}$, R $= \frac{1}{a}$, T $=$ N $= \frac{1}{y}\sqrt{1 + \frac{y^2}{a^2}}$, S $= \frac{1}{y}$ and the fluent $\frac{d}{n} x^n\text{N} + \frac{e}{rf} d\text{R} \left| \frac{\text{R}+\text{T}}{\text{S}} \right.$ $=$

$$= \tfrac{1}{2} y^2 N + \tfrac{1}{2} a \left| \frac{\frac{1}{a} + N}{\frac{1}{y}} \right. = \tfrac{1}{2} y^2 N + \tfrac{1}{2} a \left| \frac{\frac{y^2}{a} + y^2 N}{y} \right. = \text{SWP}$$ but by art. 122. $\text{ST} = \frac{y^2}{a}$, and $a = \frac{y^2}{\text{ST}}$, and $\text{PT} = \sqrt{y^2 + \frac{y^4}{a^2}} = y^2 N$, therefore $\text{SWP} = \tfrac{1}{2}\,\text{PT} + \frac{\text{SP}^2}{2\text{ST}} \left| \frac{\text{ST} + \text{PT}}{\text{SP}} \right.$ *.

267. If $m = -1$, then the fluxion of $\text{SWP} = \dot{y}\sqrt{1 + \frac{a^2}{y^2}}$ which may be compared with the fluxion in art. 94. Exam. 8. where $d=1$, $\eta = -2$, $e=1$, $f=a^2$, $\text{R}=a$, $\text{T} = \text{N} = \sqrt{a^2 + y^2} =$ (art. 122.) $\sqrt{\text{ST}^2 + \text{SP}^2} = \text{PT}$, $\text{S}=y$, and the fluent is $\frac{-2d}{\eta}\text{N} + \frac{2d}{\eta}\text{R} \left| \frac{\text{R}+\text{T}}{\text{S}} \right. = \text{PT} - \text{ST} \left| \frac{\text{ST}+\text{PT}}{\text{SP}} \right. + \text{C}$. Suppose it to vanish when $\text{SP} = \text{SW}$ and $\text{PT} = \text{WG}$ then $\text{C} = -\text{WG} + \text{ST} \left| \frac{\text{SG}+\text{WG}}{\text{SW}} \right.$ and the arc $\text{WP} = \text{PT} - \text{WG} - \text{ST} \left| \frac{\text{ST}+\text{PT}}{\text{SP}} \right. + \text{ST} \left| \frac{\text{SG}+\text{WG}}{\text{SW}} \right.$ †.

* Cotes Harmon. Mensur. p. 22.
† Ibid. p. 23.

268. In the equiangular spiral we have $a\dot{y} = y\dot{x}$ (art. 124.) and $\frac{y\dot{x}}{r} = \frac{a\dot{y}}{r}$ therefore $\dot{y}^2 + \frac{y^2\dot{x}^2}{r^2} = \dot{y}^2 \times \frac{a^2+r^2}{r^2}$ and $\sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{r^2}} = \frac{\dot{y}\sqrt{a^2+r^2}}{r}$ the fluent of which is $\frac{y\sqrt{a^2+r^2}}{r}$, which is equal to the curve SP contained between S and P after an infinite number of revolutions of the ray SP, therefore the arc SP : ray SP :: $\sqrt{a^2+r^2} : r$ :: $\sqrt{\frac{a^2y^2}{r^2} + y^2} : y$ :: PT : SP and arc SP $=$ PT.

PROP. XLV.

269. *To find the length of the curve when the curve is defined by an equation between the right lines* SP, FP *drawn to the two points* S *and* F. (*fig.* 70.)

Call SP, z, FP, v, SF, $2a$, the arc of a circle whose radius is r subtending the angle ASP call x, and the arc AP, t, then by Prop. XXII. $SY^2 = \frac{z^4\dot{x}^2}{r^2\dot{t}^2}$, but $\frac{z^2\dot{x}^2}{r^2} = \dot{t}^2 - \dot{z}^2$ therefore

$SY^2 = z^2 - \frac{z^2\dot{z}^2}{\dot{t}^2}$ and $\dot{t}^2 = \frac{z^2\dot{z}^2}{z^2 - SY^2}$ therefore

$$\dot{t} = \frac{z\dot{z}}{\sqrt{z^2 - SY^2}}.$$

Corol. Since $SY = \frac{z^2\dot{x}}{r\dot{t}}$ we have $\frac{1}{2} SY \times \dot{t} = \frac{z^2\dot{x}}{2r} =$ the fluxion of the area SAP by Prop. XLI.

 Example.

Example.

Let $z+v=2b$, put $b^2-a^2=m^2$, then by art 127, $SY^2=\frac{m^2z}{v}$, and therefore $\dot{t}=$

$$\frac{z\dot{z}}{\sqrt{z^2-\frac{m^2z}{v}}}=\frac{\dot{z}\sqrt{zv}}{\sqrt{vz-m^2}}$$ the fluent of which is equal to the arc AP. And since $SY=\frac{mz}{\sqrt{zv}}$ we have $\dot{\overline{ASP}}=\frac{mz\dot{z}}{2\sqrt{vz-m^2}}$.

SECT.

SECT. X.

Of the Contents of SOLIDS.

Lemma VII.

270. *Let* BDEC (*fig.* 68.) *be a Trapezium, whose angles at* B *and* C *are right ones, and let* BHKC *be a rectangle whose height is* BH, *draw* DF *parallel to* BC, *and let* BC *be infinitely small, then the ultimate ratio of the solid generated by the Trapezium* BDEC *is to the cylinder generated by the rectangle* BHKC *as* $\overline{BD + \frac{1}{2} FE}|^2 : BH^2$.

Let p be the area of a circle whose radius is 1, then the cone ACE $= \frac{1}{3} \times p \times CE^2 \times AC$, and the cone ABD $= \frac{1}{3} p \times BD^2 \times AB$, therefore the frustrum BDEC $= \frac{1}{3} p \times$ $\overline{\overline{CE^2 - BD^2} \times AB + CE^2 \times BC}$, but AB $=$ $\frac{BC \times BD}{CE - BD}$ therefore the frustrum BDEC $= \frac{1}{3}$ $p \times \overline{BC \times BD \times \frac{CE^2 - BD^2}{CE - BD} + CE^2} = \frac{1}{3} p \times$ $\overline{BC \times BD \times \overline{CE + BD} + CE^2} = \frac{1}{3} p \times BC \times$ $BD \times \overline{2BD + EF} + \overline{BD + EF}|^2 = \frac{1}{3} p \times BC \times$ $3BD^2 + 3BD \times EF + EF^2$. But the cylinder BHKC $= p \times BC \times BH^2$, therefore the frustrum BDEC is to the cylinder BHKC as $BD^2 + BD \times EF + \frac{1}{3} EF^2$ to BH^2 that is, when BC and EF are

infinitely small as $BD^2 + BD \times EF : BH^2$ but $\overline{BD + \frac{1}{2}EF}|^2 = BD^2 + BD \times EF + \frac{1}{4}EF^2 = BD^2 + BD \times EF$, therefore the frustrum BDEC : cylinder BHKC :: $\overline{BD + \frac{1}{2}EF}|^2 : BH^2$. Q. E. D.

Prop. XLVI.

271. Let AP (fig. 58.) be any curve, AO its abscissa OP an ordinate, let $AO = x$, $OP = y$, and let p be the area of a circle whose radius is 1, then the fluxion of the solid generated by AP revolving round the axis AO is equal to $py^2\dot{x}$.

For supposing the same construction as in prop. 39, draw pq parallel to AO, then by Lemma VII. the solid generated by $op\pi\omega$ is to the cylinder generated by $ok\kappa\omega$ as $\overline{op + \frac{1}{2}\pi q}|^2 : AL^2 :: OP^2 : AL^2$, that is, the fluxion of the solid AOP is to the fluxion of the cylinder ALKO as $OP^2 : AL^2 :: y^2 : a^2$, and the fluxion of the solid $AOP = \frac{y^2}{a^2} \times \dot{\overline{ALKO}}$, but $ALKO = pa^2x$, and $\dot{\overline{ALKO}} = pa^2\dot{x}$, therefore the fluxion of the solid AOP is equal to $pa^2\dot{x} \times \frac{y^2}{a^2} = py^2\dot{x}$. Q. E. D.

Corol. 1. Therefore to find the solid generated by AOP compute $py^2\dot{x}$ and its fluent will be the content of the solid.

Corol. 2. Since $py^2\dot{x} = pb^2 \times \frac{y^2\dot{x}}{b^2}$ the solid AOP will be equal to a cylinder the radius of whose base is b and whose altitude is the fluent of $\frac{y^2\dot{x}}{b^2}$.

Example

Example I. *Fig.* 5. 13.

272. *To find the content of the solid generated by the revolution of a parabola or hyperbola whose equation is* $y = ax^m$, *about the abscissa* AO.

By this equation we have $y^2 = a^2 x^{2m}$ and $py^2\dot{x} = pa^2\dot{x}x^{2m}$, the fluent of which is $\frac{pa^2 x^{2m+1}}{2m+1} + C$.

If the curve is a parabola, this fluent is to vanish when $x = 0$, and the solid will be $= \frac{pa^2x^{2m+1}}{2m+1} = \frac{py^2x}{2m+1} = \frac{1}{2m+1} \times$ the cylinder generated by the rectangle AOPM.

In the common parabola $m = \frac{1}{2}$ and the solid is equal to half the circumscribing cylinder.

If the curve is an hyperbola, then if $-2m$ is less than 1, or $-m$ less than $\frac{1}{2}$, the fluent will vanish when $x = 0$, and the solid AOP $= \frac{1}{2m+1} \times$ cylinder AOPM, and the solid OPH will be infinite.

If $-m$ is greater than $\frac{1}{2}$, then the fluent will vanish when x is infinite, and the solid OPH $= \frac{1}{2m+1} \times$ cylinder AOPM, and the solid AOP infinite.

In the common hyperbola, $m = -1$, and the solid OPH will be equal to the cylinder generated by the rectangle AOPM.

If $m = -\frac{1}{2}$, then the fluxion of the solid will be $\frac{pa^2\dot{x}}{x}$ whose fluent which vanishes when $x=b$ is $pa^2 \left| \frac{x}{b} \right. = py^2x \left| \frac{x}{b} \right.$. Therefore if AO (fig. 59.) $= x$, AB$=b$, the solid generated by BEOP will $= py^2x \left| \frac{x}{b} \right.$ and this solid is to a cylinder the diameter of whose base is OP and altitude AO as 1 to the measure of the ratio between AO and AB, the modulus being 1.

Example II. *Fig.* 6.

273. *To find the content of the spheroid generated by the revolution of the ellipse* APB *round the axis* AB.

Call AB, $\frac{a}{b}$, the latus rectum a, AO, x, OP, y, then $y^2 = ax - bxx$ and $py^2\dot{x} = pax\dot{x} - pbx^2\dot{x}$ the fluent of which is $\frac{1}{2}pax^2 - \frac{1}{3}pbx^3 =$ to the solid AOP. When $x = \frac{a}{b}$ this solid will be equal to $\frac{pa^3}{2b^2} - \frac{pa^3}{3b^2} = \frac{pa^3}{6b^2} = \frac{4p}{6} \times \frac{a}{b} \times \frac{a^2}{4b} = \frac{2}{3} \times p \times \text{AB} \times \text{SG}^2 = \frac{2}{3}$ of the circumscribing cylinder.

If $b = 1$, the solid will be a sphere whose content is $\frac{2}{3}$ of the circumscribing cylinder.

If b is negative the curve will be an hyperbola, (fig. 7.) and the solid AOP $= \frac{1}{2}pax^2 + \frac{1}{3}pbx^3$.

Example

Example III. *Fig.* 72.

274. *To find the content of the solid generated by the Cissoid of Diocles, revolving round the diameter of its generating circle.*

Call AB, a, AO, x, OP, y, then $y^2 = \frac{x^3}{a-x}$ which by division becomes $-x^2 - ax - a^2 - \frac{a^3}{a-x}$ and $py^2\dot{x} = p\dot{x} \times -x^2 - ax - a^2 - \frac{a^3}{a-x}$ the fluent of which is $p \times -\frac{1}{3}x^3 - \frac{1}{2}ax^2 - a^2x - a^3 \Big| \frac{a-x}{a} = p \times a^3 \Big| \frac{a}{a-x} - a^2x - \frac{1}{2}ax^2 - \frac{1}{3}x^3 = \frac{px^3}{a-x}$ into $\frac{a^4 - a^3x^3}{x^3} \Big| \frac{a}{a-x} - \frac{a^3 - a^2x}{x^2} - \frac{a^2 - ax}{2x} - \frac{a-x}{3}$.

Let AO, AB, AR, AS, AT, be taken in continual proportion then $AR = \frac{a^2}{x}$, $AS = \frac{a^3}{x^2}$, $AT = \frac{a^4}{x^3}$ and the solid AOP is equal to a cylinder whose base is a circle described by the radius OP, and whose altitude is $ST \Big| \frac{AB}{BO} - RS - \frac{1}{2}BR - \frac{1}{3}BO$. Q. E. I *.

* Cotes Harmon. Mensur. p. 24.

Lemma

Lemma VIII.

275. *If* SV (*fig.* 71.) *be an arc of a circle whose radius is* FS *and versed sine* VW, *then the solid described by the sector* SFV *revolving round the axis* FV *is equal to* $\frac{2}{3} \times p \times \mathrm{FV}^2 \times \mathrm{VW}$.

For by art. 273, the solid $\mathrm{VSW} = p \times \mathrm{FV} \times \mathrm{VW}^2 - \frac{1}{3}\mathrm{VW}^3$, but the cone $\mathrm{SWF} = \frac{1}{3}p \times \mathrm{FW} \times \mathrm{SW}^2 = \frac{1}{3}p \times \overline{\mathrm{FV} - \mathrm{VW}} \times \mathrm{WV} \times \overline{2\mathrm{FV} - \mathrm{WV}} = \frac{2}{3}p \times \mathrm{FV}^2 \times \mathrm{WV} - p \times \mathrm{FV} \times \mathrm{WV}^2 + \frac{1}{3}p \times \mathrm{WV}^3$ and their sum $\mathrm{SFV} = \frac{2}{3}p \times \mathrm{FV}^2 \times \mathrm{WV}$. Q. E. D.

Example IV. *Fig.* 63.

276. *To find the content of the solid generated by the* Conchoid *of* Nicomedes *revolving round the axis* AG.

Call AG, a, GF, b, GO, x, OP, y, then $y^2 = \frac{\overline{b+x}|^2}{x^2} \times \overline{a^2 - x^2}$, and the fluxion of the solid $\frac{\mathrm{AOP}}{p} = -y^2\dot{x} = \frac{-a^2\dot{x} \times \overline{b+x}|^2}{x^2} + \dot{x} \times \overline{b+x}|^2$ $= \dot{x} \times \overline{b+x}|^2 - \frac{a^2b^2\dot{x}}{x^2} - \frac{2a^2b\dot{x}}{x} - a^2\dot{x}$, the fluent of which is $\frac{\overline{b+x}|^3}{3} + \frac{a^2b^2}{x} + 2a^2b \,|\frac{a}{x} - a^2x + \mathrm{C}$, which is to vanish when $x = a$, therefore C $= -$

$= -\frac{\overline{b+a}|^3}{3} - ab^2 + a^3 = -\frac{1}{3}b^3 - 2ab^2 - a^2b + \frac{2}{3}a^3$, and the solid $\frac{AOP}{p} = \frac{\overline{b+x}|^3}{3} + \frac{a^2b^2}{x} - a^2x - \frac{1}{3}b^3 - 2ab^2 - a^2b + \frac{2}{3}a^3 + 2a^2b \left| \frac{a}{x} \right.$.

The cone described by the triangle OPF is equal to $\frac{1}{3}p \times OP^2 \times OF$; and $\frac{OPF}{p} =$
$\frac{y^2 \times \overline{b+x}}{3} = \frac{\overline{b+x}|^3 \times \overline{a^2 - x^2}}{3x^2} = \frac{a^2 \times \overline{b+x}|^3}{3x^2} - \frac{\overline{b+x}|^3}{3} = \frac{a^2b^3}{3x^2} + \frac{a^2b^2}{x} + a^2b^3 + \frac{a^2x}{3} - \frac{\overline{b+x}|^3}{3}$.

Also the cone $\frac{FGR}{p}$ is equal to $\frac{1}{3} \times GR^2 \times FG = \frac{b \times b^2 \times \overline{a^2 - x^2}}{3x^2} = \frac{a^2b^3}{3x^2} - \frac{b^3}{3}$, therefore the solid $\frac{OGRP}{p}$ is equal to $-\frac{\overline{b+x}|^3}{3} + \frac{a^2b^2}{x} + a^2b + \frac{b^3}{3} + \frac{a^2x}{3}$ which added to the solid $\frac{AOP}{p}$ we have the solid $\frac{AGRP}{p} = \frac{2a^2b^2}{x} - \frac{2a^2x}{3} - 2ab^2 + \frac{2a^3}{3} + 2a^2b \left| \frac{a}{x} \right. = \overline{1 + \frac{3b^2}{ax}} \times \frac{2a^2}{3} \times \overline{a - x} + 2a^2b \left| \frac{a}{x} \right.$.

With the center F and radius FV = AG, describe the circular arc VS meeting FP in S and let fall the perpendicular SW on FV, then since the triangles FWS, PQR are similiar and FS = PR, we

we have $FW = PQ = x$, and by Lemma VIII. the solid described by the sector FVS is equal to $\frac{2}{3}p \times a^2 \times \overline{a - x}$, therefore the solid AGRP is equal to $\overline{1 + \frac{3b^2}{ax}} \times FVS + 2pa^2b \left| \frac{a}{x} \right. =$ $\frac{PR \times GO + 3FG^2}{RP \times GO} \times FVS + 2p \times AG^2 \times FG \left| \frac{RP}{GO} \right.$ or since $FR : FG :: RP : GO$ we have $GO = \frac{FG \times RP}{FR}$ and the solid AGRP $=$ $\frac{RP^2 + 3FG \times FR}{RP^2} \times FVS +$ a cylinder the radius of whose base is AG, and whose altitude is $2FG \left| \frac{FR}{FG} \right.$. Q. E. I *.

Example V. *Fig.* 15.

277. *To find the content of the solid generated by the revolution of the logarithmic curve about its asymptote.*

Call the subtangent OT, a, OP, y, AO, x, then $\dot{x}y = a\dot{y}$ and $p\dot{x}y^2 = pay\dot{y}$ whose fluent is $\frac{1}{2}pay^2$, which vanishes when $y = 0$, therefore the solid OPB infinite in length is equal to $\frac{1}{2}payy$, that is, equal to a cylinder that has the same base as the solid OPB, and whose altitude is half the subtangent OT. Q. E. I.

* Cotes Harmon. Mensur. p. 25.

Example VI. *Fig.* 25.

278. *To find the content of the solid generated by the revolution of the quadratrix* AOP *about the axis* AG.

Supposing the same things as in art. 218. we have $\dot{x} = -\dot{y} \times \overline{\frac{2y}{3r} + \frac{4y^3}{45r^3} + \frac{4y^5}{315r^5}}$ &c. and therefore the fluxion of the solid AOP or $-$ $py^2\dot{x} = p\dot{y} \times \overline{\frac{2y^3}{3r} + \frac{4y^5}{45r^3} + \frac{4y^7}{315r^5}}$ &c. the fluent of which is $p \times \overline{\frac{y^4}{6r} + \frac{2y^6}{135r^3} + \frac{y^8}{630r^5}}$ &c. $=$ a cylinder whose base is the circle described by OP, and its altitude is $\frac{y^2}{6r} + \frac{2y^4}{135r^3} + \frac{y^6}{630r^5}$ &c. Q. E. I.

Sect. XI.

Of the Quadrature of Curve Surfaces.

Lemma IX.

279. *Let* ABC (*fig.* 74.) *be a cone whose base is the circle* BC, *and altitude* AE, *I say the surface of the cone is equal to the circumference of the base multiplied by half the side* AB.

For let the two right lines AD, A*d* be drawn from the vertex to the circumference of the base, infinitely near each other, then the triangle AD*d* $= \frac{1}{2}$ AD $\times$ D*d* will be the increment of the surface ABD, call the side AB, r, the arc BD, z, then the fluxion of the surface will be equal to $\frac{1}{2}r\dot{z}$, and taking the fluents, the surface ABD $= \frac{1}{2}rz = \frac{1}{2}$AB $\times$ BD, and the whole surface $= \frac{1}{2}$AB multiplied into the circumference of the circle BC. Q. E. D.

280. *Corol.* 1. Let BCED (fig. 68.) be a trapezium right angled at B and C, let DF be parallel to BC, and p be the area of a circle whose radius is 1, then the surface described by DE revolving round the axis BC is equal to $2p \times$ DE $\times$ $\overline{\text{BD} + \frac{1}{2}\text{EF}}$, for let ED, CB be produced till they meet in A, then the surface described by AE is equal to the circumference of a circle whose radius is CE multiplied into $\frac{1}{2}$AE, that is, equal to $p \times$ AE $\times$ EC, and the surface described by AD is equal to $p \times$ AD$\times$DB, and therefore the surface described by DE is equal to $p \times \overline{\text{AE}\times\text{EC} - \text{AD}\times\text{DB}}$.

But

But because the triangles ADB, AEC, DEF are similiar, $AD = \frac{DB \times DE}{EF}$, and $AE = \frac{CE \times DE}{EF}$, therefore the surface $DE = p \times DE \times \frac{CE^2 - DB^2}{EF} = p \times DE \times \frac{CE^2 - DB^2}{CE - DB} = p \times DE \times \overline{CE + DB} = 2p \times DE \times \overline{BD + \frac{1}{2} EF}$.

Corol. 2. If DF is parallel to BC, the surface described by $DF = 2p \times DF \times BD$.

PROP. XLVII.

281. *Let* AP *(fig.* 58.*) be any curve,* AO *the abscissa,* OP *an ordinate, let p be the area of a circle whose radius is* 1, *call* AO, *x*, OP, *y*, *and the arc* AP, *t*, *then the fluxion of the surface described by* AP *is equal to* $2py\dot{t}$.

For supposing the same construction as in Prop. XXXIX. the surface described by $p\pi = 2p \times p\pi \times \overline{op + \frac{1}{2}\pi q} = 2p \times p\pi \times OP$ (by Lemma 9. Cor. 1.) and the surface described by $k\kappa = 2p \times k\kappa \times AL$, therefore as $k\kappa \times AL : p\pi \times OP ::$ surface $k\kappa$: surface $p\pi$ so is the fluxion of the surface LK to the fluxion of the surface AP, and the fluxion of the surface $AP = \frac{p\pi \times OP}{k\kappa \times AL} \times \overline{\text{surface LK}} = \frac{y\dot{t}}{a\dot{x}} \times \overline{\text{surface LK}}$. But the surface $LK = 2pax$, and its fluxion $= 2pa\dot{x}$, therefore the fluxion of the surface $AP = 2py\dot{t} = 2py\sqrt{\dot{x}^2 + \dot{y}^2}$. Q. E. D.

Corol.

Corol. 1. The surface AP is equal to the fluent of $\overline{2py\sqrt{\dot{x}^2+\dot{y}^2}}$.

Corol. 2. The surface AP is equal to a circle, the square of whose radius is equal to the fluent of $\overline{2y\sqrt{\dot{x}^2+\dot{y}^2}}$.

Example I.

282. *To measure the surface of the solid generated by the revolution of a parabola or hyperbola whose equation is $y=ax^m$, round the axis* AO.

By art. 232. we have $\dot{t}=\dot{x}\sqrt{1+m^2a^2x^{2m-2}}=\dot{x}x^{m-1}\sqrt{m^2a^2+x^{2-2m}}$ and $2py\dot{t}=2pa\dot{x}x^{2m-1}\sqrt{m^2a^2+x^{2-2m}}$, the fluent of which is equal to the surface AP.

283. In the common parabola, $m=\frac{1}{2}$, and the fluxion of the surface is $2pa\dot{x}\sqrt{\frac{1}{4}a^2+x}$ the fluent of which by art. 51, is $\frac{4ap}{3}\times\overline{\frac{1}{4}a^2+x}|^{\frac{3}{2}}+C$ which is to vanish when $x=0$, therefore $C=-\frac{4ap}{3}\times\frac{a^3}{8}=\frac{-pa_4}{6}$, and the surface AP $=\frac{4ap}{3}\times\overline{\frac{1}{4}a^2+x}|^{\frac{3}{2}}-\frac{pa^4}{6}$.

Let S (fig. 75.) be the focus then $\frac{1}{4}a^2=$ AS, take AQ $=$ SP $=$ AS $+$ AO $=\frac{1}{4}a^2+x$, then RQ $=\sqrt{a^2\times\overline{\frac{1}{4}a^2+x}}=a\sqrt{\frac{1}{4}a^2+x}$, and the surface AP $=\frac{4P}{3}\times$ AQ $\times$ RQ $-\frac{8p}{3}\times$ AS^2. But the area ARQ $=\frac{2}{3}\times$ AQ $\times$ RQ, and the area ASM $=\frac{2}{3}$ AS $\times$ SM $=\frac{4}{3}\times$ AS^2, therefore the surface AP $=2p\times\overline{\text{ARQ}-\text{ASM}}=2p\times$ SMQR.

284.

284. In the common hyperbola (fig. 76.) $m = -1$, and the fluxion of the surface is $2pa\dot{x}x^{-3}\sqrt{a^2+x^4} = 2pa\dot{x}x^{-1}\sqrt{\frac{a^2}{x^4}+1}$ which may be compared with the fluxion in art. 94. Exam. 5. where $d=2ap$, $n=-4$, $e=1$, $f=a^2$, $R=1$, $N=T=\sqrt{1+\frac{a^2}{x^4}} = \frac{1}{x}\sqrt{x^2+\frac{a^2}{x^2}} = \frac{1}{x}\sqrt{x^2+y^2}$ $S=\frac{a}{x^2}$, and the fluent $\frac{2d}{n}N - \frac{2d}{n}R\left|\frac{R+T}{S}\right. = \frac{-pa}{x}\sqrt{x^2+y^2}+pa\left|\frac{x^2+x\sqrt{x^2+y^2}}{a}\right.+C$, suppose it to vanish when $x=AB=b$ then $C = \frac{pa}{b}\sqrt{b^2+BC^2} - pa\left|\frac{b^2+b\sqrt{b^2+BC^2}}{a}\right.$ and the whole fluent is $\frac{pa}{b}\sqrt{b^2+BC^2} - \frac{pa}{x}\sqrt{x^2+y^2}$ $+pa\left|\frac{x^2+x\sqrt{x^2+y^2}}{b^2+b\sqrt{b^2+BC^2}}\right. = \frac{pa}{b}\times AC - \frac{pa}{x}\times AP$ $+pa\left|\frac{\frac{x^2}{b}+\frac{x}{b}\times AP}{b+AC}\right.$. Through C draw CG parallel to AP then $GB = \frac{BC\times AO}{OP} = \frac{\frac{a}{b}\times x}{\frac{a}{x}} = \frac{x^2}{b}$, and $GC = \frac{BC\times AP}{OP} =$

$\frac{x}{b} \times AP$. Let AP cut BC in F then $AP = \frac{AO \times AF}{AB}$ and $\frac{a}{x} \times AP = BC \times AF$, and the surface described by the arc $CP = p \times BC \times AC - p \times BC \times AF + p \times AB \times BC \Big| \frac{GB + GC}{AB + AC}$. But the circle whose radius is BC is equal $p \times BC^2$ therefore the surface described by CP is to the circle whose radius is BC as $AC - AF + AB \Big| \frac{GB + GC}{AB + AC}$ to BC*.

Example II. *Fig.* 1.

285. *To measure the surface of a sphere.*

Call the diameter AB, a, AO, x, OP, y, then $ax - xx = yy$ and $2y\dot{y} = \dot{x} \times \overline{a - 2x}$, therefore $\dot{y}^2 = \dot{x}^2 \times \frac{a^2 - 4ax + 4xx}{4y^2}$ and $\dot{x}^2 + \dot{y}^2 = \frac{a^2\dot{x}^2}{4y^2}$, therefore $2py\sqrt{\dot{x}^2 + \dot{y}} = pa\dot{x}$, the fluent of which is $pax = p \times AP^2$, therefore the surface described by AP is equal to a circle whose radius is the chord AP. Q. E. I.

Corol. The whole surface of the sphere is equal to a circle whose radius is AB, that is, equal to four times the circle whose radius is AC.

* Harm. Mensur. p. 94.

Example

Example III.

To measure the surface described by an Hyperbola, revolving round one of its axes.

286. Case 1. Suppose the axis round which the hyperbola revolves to be the transverse axis CA (fig. 77.)

Call CA, a, AB, b, CO, x, OP, y, and put $\frac{a^2+b^2}{a^2}=c^2$, and by art. 257, we have $\sqrt{\dot{x}^2+\dot{y}^2}=$ $\frac{\dot{x}\sqrt{c^2x^2-a^2}}{\sqrt{x^2-a^2}}=\frac{b\dot{x}\sqrt{c^2x^2-a^2}}{ay}$ and the surface will be equal to a circle, the square of whose radius is equal to the fluent of $2y\sqrt{\dot{x}^2+\dot{y}^2}=$ $\frac{2b\dot{x}}{a}\sqrt{c^2x^2-a^2}$ which may be compared with the fluxion in art. 94. Exam. 6. where $d=\frac{2b}{a}$, $\eta=2$, $e=-a^2$, $f=c^2$, $\mathrm{R}=c$, $\mathrm{N}=\mathrm{T}=$ $\frac{1}{x}\sqrt{c^2x^2-a^2}$, $\mathrm{S}=\frac{a}{x}\sqrt{-1}$, and the fluent $\frac{d}{\eta}x^\eta\mathrm{N}$ $+\frac{e}{\eta f}d\mathrm{R}\left|\frac{\mathrm{R}+\mathrm{T}}{\mathrm{S}}\right.=\frac{b}{a}x^2\mathrm{N}-\frac{ab}{c}\left|\frac{c+\mathrm{N}}{\frac{a}{x}\sqrt{-1}}\right.+\mathrm{C}$ which is to vanish when $x=a$, and $\mathrm{N}=$ $\frac{1}{a}\sqrt{c^2a^2-a^2}=\frac{b}{a}$, therefore $\mathrm{C}=-b^2$ $+\frac{ab}{c}\left|\frac{c+\frac{b}{a}}{\sqrt{-1}}\right.$ and the correct fluent is b into $\frac{x^2\mathrm{N}}{a}$

$$-b-\frac{a}{c}\left|\frac{c+N}{\frac{ac}{x}+\frac{b}{x}}\right.=b \text{ into } \frac{x^2N}{a}-b$$

$$-\frac{a}{c}\left|\frac{x+\frac{xN}{c}}{a+\frac{b}{c}}\right.,$$

Let F be the focus, CB the asymptote, then $CF = \sqrt{a^2+b^2} = ac$. Let CF : CA :: CA : CE, then $CE = \frac{a}{c}$ and $EG = \frac{CE \times AB}{CA} = \frac{b}{c}$ and $CG = a$. In the angle CEG inscribe the right line CZ = CO and produce it to V meeting OP produced in V, then $EZ = \sqrt{CZ^2 - CE^2} = \sqrt{x^2 - \frac{a^2}{c^2}} = \frac{1}{c}\sqrt{c^2x^2 - a^2} = \frac{xN}{c}$, and $OV = \frac{CO \times EZ}{CE} = \frac{x^2N}{a}$. Therefore the surface AP is equal to a circle whose radius is a mean proportional between AB and $OV - AB - CE \left|\frac{CZ+ZE}{CG+GE}\right.$. And therefore the surface AP is to the circle whose radius is AB as $OV - AB - CE \left|\frac{CZ+ZE}{CG+GE}\right.$ to AB *

287. Case 2. Let the axis round which the hyperbola revolves be the conjugate axis CD perpendicular to AC at C (fig. 78.) call CO, x, OP, y,

* Cotes Harm. Mensur. p. 26.

CA,

CA, a, CD, b, and let $\frac{a^2+b^2}{b^2}=c^2$, then $x^2=\frac{b^2}{a^2}\times\overline{y^2-c^2}$ or $a^2x^2=b^2y^2-a^2b^2$ and $y^2=\frac{a^2}{b^2}\times\overline{x^2+b^2}$.

Therefore in the fluent in the last case change $-a^2$ into $+b^2$ and b^2 into $-a^2$, then $e=+b^2$, $N=\frac{1}{x}\sqrt{c^2x^2+b^2}$ and the surface is equal to

$$\frac{ax^2N}{b}+\frac{ab}{c}\Big|\frac{c+N}{\frac{b}{x}}=a \text{ into } \frac{x^2N}{b}+\frac{b}{c}\Big|\frac{x+\frac{xN}{c}}{\frac{b}{c}}.$$

Let F be the focus, then $CF=\sqrt{a^2+b^2}=bc$, let CF : CD :: CD : CE, then $CE=\frac{b}{c}$. join EO then $EO=\sqrt{x^2+\frac{b^2}{c^2}}=\frac{1}{c}\sqrt{c^2x^2+b^2}=\frac{xN}{c}$.

Therefore the surface described by AP is equal to a circle whose radius is a mean proportional between CA and $\frac{CO\times EO}{CE}+CE\Big|\frac{CO+EO}{CE}$, and the surface AP is to the circle whose radius is CA as $\frac{CO\times EO}{CE}+CE\Big|\frac{CO+OE}{CE}$ to CA.

Example IV. *Fig.* 79.

To measure the surface of a spheroid.

288. Case I. Let the axis of the spheroid be the conjugate axis CD of the generating ellipse. Call AC, a, CD, b, CO, x, OP, y, then $x^2 = \frac{b^2}{a^2} \times \overline{a^2 - y^2}$ and $y^2 = \frac{a^2}{b^2} \times \overline{b^2 - x^2} = \frac{-a^2}{b^2} \times \overline{x^2 - b^2}$, therefore in the fluent in the second case of the last example change b^2 into $-b^2$, then $-c^2 = \frac{a^2 - b^2}{b^2}$, put $d^2 = -c^2$ and $N\sqrt{-1} = \frac{1}{x}\sqrt{d^2x^2 + b^2}$, and the surface will be equal a into $\frac{x^2N}{b\sqrt{-1}} +$

$$\frac{b}{d}\left|\frac{x + \frac{xN}{c}}{\frac{b}{d}}\right. = a \text{ into } \frac{x^2N\sqrt{-1}}{b} + \frac{b}{d}\left|\frac{x + \frac{xN\sqrt{-1}}{d}}{\frac{b}{d}}\right..$$

Let F be the focus then CF $= bd$, let CE $= \frac{\text{CD}^2}{\text{CF}} = \frac{b}{d}$, join EO then EO $= \sqrt{x^2 + \frac{b^2}{d^2}} = \frac{x}{d} N\sqrt{-1}$, and the surface AP is equal to a circle whose radius is a mean proportional between AC and $\frac{\text{CO} \times \text{EO}}{\text{CE}} + \text{CE}\left|\frac{\text{CO} + \text{EO}}{\text{CE}}\right.$. And this surface is to

to the circle whose radius is AC as $\frac{CO \times EO}{CE} + CE\left|\frac{CO + EO}{CE}\right.$ to AC.

289. Case II. Let the axis of the spheroid be AC the tranverse axis of the ellipse, call AC, a, CD, b, Co, x, oP, y, then $y^2 = \frac{b^2}{a^2} \times \overline{a^2 - x^2}$, therefore in the fluent of the first case change a into b and b into a then $-c^2 = \frac{b^2 - a^2}{a^2} = -\frac{a^2 - b^2}{a^2}$ therefore the sign of c^2 or R^2 will be changed, and the measure $R\left|\frac{R+T}{S}\right.$ (by art. 234.) will be changed into the measure of an angle whose radius, tangent, and secant are as R, T, and S, or c, $\frac{1}{x}\sqrt{a^2 - c^2x^2}$ and $\frac{a}{x}$, or as x, $\sqrt{\frac{a^2}{c^2} - x^2}$ and $\frac{a}{c}$ that is, (if the right line $Ce = \frac{CA^2}{CF} = \frac{a}{c}$ be inscribed in the angle Coe) the measure of the angle eCo and the fluent will be equal to a mean proportional between CD and $\frac{Co \times oe}{Ce} + Ce\,(eCo)$ which vanishes when $x = 0$, therefore the surface described by the arc DP is to the circle whose radius is DC as $\frac{Co \times oe}{Ce} + Ce\,(eCo)$ to DC.

Example V. *Fig.* 72.

290. *To measure the ſurface generated by the Ciſſoid of* Diocles *revolving round the Aſymptote* BE.

Call AB a, BQ = OP, x, PQ = OP, y, then $x^2 = \frac{\overline{a-y}^3}{y}$ and by art. 258. $\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{1}{2}a\dot{y}y^{-\frac{3}{2}}\sqrt{a+3y}$ and $2y\sqrt{\dot{x}^2+\dot{y}^2} = a\dot{y}y^{-\frac{1}{2}}\sqrt{a+3y}$ which may be compared with the fluxion in art. 94. Exam. 6. where $d = a$, $\eta = 1$, $e = a$, $f = 3$, $R = \sqrt{3}$, $N = T = \sqrt{\frac{a+3y}{y}}$, $S = \sqrt{\frac{a}{y}}$ and the fluent $\frac{d}{\eta}x^{\eta}N + \frac{e}{\eta f}dR \left| \frac{R+T}{S} \right. =$

$$ayN + \frac{a^2}{\sqrt{3}}\left| \frac{\sqrt{3}+N}{\sqrt{\frac{a}{y}}} \right. = a \text{ into}$$

$$yN + \frac{a}{\sqrt{3}}\left| \frac{\sqrt{ay} + N\sqrt{\frac{1}{3}ay}}{a\sqrt{\frac{1}{3}}} \right.$$ which vaniſhes when $y = 0$ therefore the ſurface deſcribed by the infinitely long arc PZ is equal to a circle whoſe radius is a mean proportional between a and $yN + \frac{a}{\sqrt{3}}\left| \frac{\sqrt{ay} + N\sqrt{\frac{1}{3}ay}}{a\sqrt{\frac{1}{3}}} \right.$.

Suppoſing the ſame conſtruction as in art. 258. we have $BF = a\sqrt{\frac{1}{3}}$, $DF = N\sqrt{\frac{1}{3}ay}$, $AE = \frac{1}{2}aN$, $BD = \sqrt{ay}$ and ſince BE and OG are parallel we have AB : BO :: AE : EG and $EG = \frac{1}{2}yN$ and the ſurface deſcribed by PZ will be to the circle whoſe

whose radius is AB as $2EG + BF \left| \frac{BD + DF}{BF} \right.$ to AB *.

Example VI. *Fig.* 15.

291. *To measure the surface generated by the revolution of the logarithmic curve about its asymptote.*

Supposing the same things as in art. 260, we have $\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{y}\sqrt{1 + \frac{a^2}{y^2}}$ and $2y\sqrt{\dot{x}^2 + \dot{y}^2} = 2y\dot{y}\sqrt{1 + \frac{a^2}{y^2}} = 2\dot{y}\sqrt{a^2 + y^2}$ which may be compared with the fluxion in art. 94. Exam. 6. where $d = 2$, $\eta = 2$, $e = a^2$, $f = 1$, $R = 1$, $T = N = \frac{1}{y}\sqrt{a^2 + y^2}$, $S = \frac{1}{y}$ and the fluent $\frac{d}{\eta} x^\eta N + \frac{e}{\eta f} dR \left| \frac{R + T}{S} \right. = y^2 N + a^2 \left| \frac{1 + N}{\frac{a}{y}} \right. = a$ into $\frac{y^2 N}{a} + a \left| \frac{\frac{y^2}{a} + \frac{y^2 N}{a}}{y} \right.$. which vanishes when $y = 0$. Produce OP to L so that $OL = OD + DP$, then since $OD = \frac{y^2}{a}$, and $DP = \sqrt{y^2 + \frac{y^4}{a^2}} = \frac{y^2 N}{a}$. The surface described by the infinite arc PB is equal to a circle whose radius is a mean proportional between OT and $PD + OT \left| \frac{OD + DP}{OP} \right.$ or between OT and PD + LM †.

* Cotes Harmon. Mensur. p. 92. † Ibid. p. 93.

PROP.

Prop. XLVIII.

292. *If* PZ (*fig.* 80.) *is a curve whose asymptote is* AZ, *then the surface described by the revolution of the infinite arc* PZ *about the asymptote* AZ *will be finite or infinite according as the area* OPZ *is finite or infinite.*

Call the abscissa AO, x, the ordinate OP, y, and the curve PZ, t, and let PT be a tangent in the point P. By Prop. XLVII. the fluxion of the surface described by PZ is equal $2py\dot{t}$, and by Prop. XXXIX. the fluxion of the area OPZ is equal $y\dot{x}$, therefore the fluxion of the surface is to the fluxion of the area as $2p\dot{t}$ to $\dot{x}$ that is as $2p \times$ PT to OT, which ratio must always be finite; for when P falls in AZ, PT is equal to OT, therefore the surface must be to the area in a finite ratio. Q. E. D. *

* Cotes Harmon. Mensur. p. 94.

Sect. XII.

Of the Centers of Gravity *and* Oscillation.

293. The Center of Gravity of one or more bodies is a point about which all the ſeveral points of theſe bodies ballance each other, and the moments on each ſide that point are equal.

Prop. XLIX.

294. *To find the Center of Gravity of any ſyſtem of bodies* A, B, C. (fig. 81.)

Let ac be a lever ſupported at S, to which are hung the weights A, B, C, by the lines aA, bB, cC, put $Sa = a$, $SB = b$, $SC = c$, let G be the Center of Gravity, call SG, g, then the moments of the bodies A, B, C, will be as their quantity of matter and velocity conjointly, that is, as $Ga \times A$, $Gb \times B$, $Gc \times C$, but the moments on each ſide of G are equal, therefore $Ga \times A = Gb \times B + Gc \times C$, or ſince $Ga = a + g$, $Gb = b - g$, $Gc = c - g$, we have $aA + gA = bB - gB + cC - gC$ and $g = \frac{bB + Cc - Aa}{A + B + C}$, but if the lever revolves about S, $bB + Ce - Aa$, is the ſum of the moments, therefore SG or g is equal to the ſum of the moments divided by the ſum of the weights.

Therefore

Therefore to find the Center of Gravity of a line, ſurface or ſolid, ſuppoſe it to be ſuſpended at the end of a lever, and find the moments of all its points, and their ſum or fluent divided by the ſum of the weights will be the diſtance of the Center of Gravity from the point of ſuſpenſion, or end of the lever.

Corol. 1. If we call the diſtance from the point of ſuſpenſion x, the weight W, the moment M then $\dot{M} = \dot{W}x$ and $CG = g = \frac{M}{W}$, and $M = gW = \left|\overline{\dot{W}x}\right|$.

Corol. 2. To find the center of gravity of the area KPAQN (fig. 82.) Let AO be a right line paſſing through the middle of the curve ſo as to biſect all the right lines PQ, pq, parallel to each other, in the points O, o, let the point of ſuſpenſion be any point S in the right line AO, call SO, x, OP, y, and let the points O, o, be infinitely near each other, then the increment of the weight is $2PO \times Oo$, and the increment of the moment $2PO \times Oo \times SO$ or ſince the fluxions are as the increments, the fluxion of the weight will be $2y\dot{x}$ which put $= 2\dot{W}$, and the fluxion of the moment $= 2\dot{M} = 2yx\dot{x}$, and then $SG = g = \frac{M}{W}$.

Corol. 3. Since $W = AKL$, we have by Corol. 1. the ſolid $AKL \times g =$ the fluent of $\dot{W}x =$ the fluent of $\overline{A\dot{O}P} \times SO$, or the ſolid $AKN \times SG =$ the fluent of $APQ \times SO$.

Corol.

Corol 4. To find the Center of Gravity of the curve line PAQ, the fluxion of the weight is $2\sqrt{\dot{x}^2+\dot{y}^2}=2\dot{W}$, and the fluxion of the moment is $2x\sqrt{\dot{x}^2+\dot{y}^2}=2\dot{M}$ and $SG=\frac{M}{W}$.

Corol. 5. Draw SB perpendicular to OS and suppose the curve PAQ to revolve about the axis SB, let p be the area of a circle whose radius is 1, then the fluxion of the surface described by AP is equal (by Prop. XLVII.) to $2px\sqrt{\dot{x}^2+\dot{y}^2}=2p\dot{M}$ and the surface described by AP equal to $2pM=$ (by Cor. 1.) $2pWg=2pg\times AP$. But $2pg$ is equal to the circumference of a circle whose radius is g, therefore the surface described by AP, is equal to the rectangle of the arc AP and the circumference described by SG, and the surface described by PAQ is equal to PAQ $\times$ the circumference described by SG.

Corol. 6. To find the Center of Gravity of the solid described by the revolution of PAQ about the axis SO, the fluxion of the weight is $py^2\dot{x}=p\dot{W}$, and the fluxion of the moment is $py^2x\dot{x}=p\dot{M}$ and $SG=\frac{M}{W}$.

Corol. 7. To find the Center of Gravity of the surface described by the revolution of PAQ about the axis SO, let $AP=t$, and the fluxion of the weight is $2py\dot{t}=2p\dot{W}$ the fluxion of the moment is $2pyx\dot{t}=2p\dot{M}$ and $SG=\frac{M}{W}$.

Examples

EXAMPLES *of finding the Center of Gravity.*

Example I. *Fig.* 83.

295. *To find the center of Gravity of the area of a parabola whose equation is* $y = ax^m$.

Suppose A to be the point of suspension, then $AO = x$, $OP = y$, $\dot{M} = yx\dot{x} = a\dot{x}x^{m+1}$, and $M = \frac{ax^{m+2}}{m+2}$, but by art. 204. $W = \frac{ax^{m+1}}{m+1}$, therefore $AG = \frac{M}{W} = \frac{m+1}{m+2} \times x$. And AG : AO :: $m+1 : m+2$, and disjointly AG : GO :: $m+1 : 1$.

In a triangle $m = 1$, and $AG = \frac{2}{3}$ AO.

In the common parabola $m = \frac{1}{2}$, and $AG = \frac{3}{5}$ AO.

In a cubical parabola, $m = \frac{1}{3}$, and $AG = \frac{4}{7}$AO.

Example II. *Fig.* 84.

296. *To find the center of gravity of the circular area* BPQD.

Call the radius CA, r; CO, x; OP, y, and suppose the point of suspension to be at C, then $\dot{M} = \dot{x}xy = -y^2\dot{y}$ and $M = -\frac{1}{3}y^3 + C$ which is

to

to vanish when $y = r$ therefore $C = \frac{1}{3}r^3$ and $M =$ $\frac{1}{3}r^3 - \frac{1}{3}y^3$ therefore $g = CG = \frac{M}{W} = \frac{r^3 - y^3}{3COPB}$.

Corol. When P comes to A then $y = 0$ and $g = \frac{r^3}{3APBC}$, but $APBC = \frac{1}{4}pr^2$ therefore $g = \frac{4r}{3p}$ and $3p : 4 :: CA : CG$ the distance of the center of gravity from the center C of the circle.

Example III. *Fig.* 62.

297. *To find the center of gravity of an ellipse,*

Call the semi-tranverse axis AC, a; the conjugate DC, b; CO, x; OP, y; about the center C with the radius CA describe the circle AMH, then $OP = \frac{b}{a} \times OM$, call the moment and weight in the circle corresponding to those in the ellipse m and w then $\dot{M} = \frac{b\dot{m}}{a}$, and $M = \frac{bm}{a}$, but by art. 211. $W = \frac{bw}{a}$ therefore $\frac{M}{W} = \frac{m}{w} = g$, that is, the center of gravity of the elliptic area PDEQ is the same as the center of gravity of the circular area M*mb*H.

Example

Example IV. *Fig.* 85.

298. *To find the center of gravity of an hyperbola.*

Call the semi-tranverse axis CA, a; the congate CD, b; CO, x; OP, y; and suppose the point of suspension to be at C, then $\frac{a^2y^2}{b^2} = x^2 - a^2$ and $\frac{a^2y\dot{y}}{b^2} = x\dot{x}$, therefore $\dot{M} = yx\dot{x} = \frac{a^2y^2\dot{y}}{b^2}$ and $M = \frac{a^2y^3}{3b^2}$ and $CG = \frac{M}{W} = \frac{a^2y^3}{3b^2W}$, therefore $CG : \frac{ay}{b} :: \frac{2ay}{b} : \frac{6W}{y}$. But by art. 212. if we make $CL = \frac{CO \times CD}{CA}$, then $W = \frac{1}{2} CO \times OP - \frac{1}{2} AC \times CD \left| \frac{CL + OP}{CD} \right.$ therefore $CG : \frac{CA \times OP}{CD} :: \frac{2CA \times OP}{CD} : 3CO - \frac{3AC \times CD}{OP} \left| \frac{CL + OP}{CD} \right.$ *.

Example V. *Fig.* 84.

299. *To find the center of gravity of the circular arc* PAQ.

Call AO, x; OP, y; AC, r; and suppose the point of suspension to be at C, then $\dot{W} =$

* Cotes Harmon. Mensur. p. 26.

4 $\sqrt{x^2 + y^2}$

$\sqrt{\dot{x}^2+\dot{y}^2} = \frac{r\dot{y}}{x}$ (art. 233.) therefore $\dot{M} = x\dot{W} = r\dot{y}$ and $M = ry$ which vanishes when $x = r$ whence $g = \frac{M}{W} = \frac{ry}{AP}$. And AP : OP : AC : C$g$.

Example VI. *Fig.* 83.

300. *To find the center of gravity of the solid generated by the revolution of a parabola whose equation is* $y = ax^m$ *round its axis* AO.

Call AO, x; OP, y; let A be the point of suspension then $\dot{M} = y^2x\dot{x} = a^2\dot{x}x^{2m+1}$, and $M = \frac{a^2x^{2m+2}}{2m+2}$ but by art. 272. $W = \frac{a^2x^{2m+1}}{2m+1}$ therefore $g = \frac{M}{W} = \frac{2m+1}{2m+2} \times x$, whence AG : AO :: $2m+1 : 2m+2$ and AG : GO :: $2m+1 : 1$.

In a cone $m = 1$, and $AG = \frac{3}{4}$ AO.

In the common parabola, $m = \frac{1}{2}$, and $AG = \frac{2}{3}$ AO.

In the cubical parabola, $m = \frac{1}{3}$, and $AG = \frac{5}{8}$AO.

Example VII. *Fig.* 84.

301. *To find the center of gravity of the segment of a sphere generated by the revolution of the circular area* AOP *about the axis* AC.

Call AC, r; CO, x; OP, y; let the point of suspension be at C, then $\dot{M} = y^2x\dot{x} = y^3\dot{y}$, and

R M =

$\dot{M} = \frac{1}{4}y^4$; which vanishes when $y=0$. Also $\dot{W} = y^2\dot{x} = r^2\dot{x} - x^2\dot{x}$, and $W = r^2x - \frac{1}{3}x^3 + C$ which is to vanish when $x=r$, therefore $C = -\frac{2}{3}r^3$ and $W = r^2x - \frac{1}{3}x^3 - \frac{2}{3}r^3 = \frac{1}{3}r^2x - \frac{1}{3}x^3 + \frac{2}{3}r^2x - \frac{2}{3}r^3 = \frac{1}{3}y^2x - \frac{2}{3}r^2 \times AO$ and $Cg = g = \frac{M}{W} = \frac{3y^4}{4y^2x - 8r^2 \times AO}$.

Corol. When O comes to C then $y=r$, and Cg the distance of the center of gravity of the hemisphere BAD from the center C is equal to $\frac{3}{8} \times AC$.

Example VIII. *Fig.* 85.

302. *To find the center of gravity of the solid generated by the revolution of an hyperbola about its axis* CA.

Call CA, a; CD, b; CO, x; OP, y; let C be the point of suspension then $\dot{M} = y^2x\dot{x} = \frac{a^2y^3\dot{y}}{b^2}$ and $M = \frac{a^2y^4}{4b^2}$. Also $\dot{W} = y^2\dot{x} = \frac{b^2x^2\dot{x}}{a^2} - b^2\dot{x}$ and $W = \frac{b^2x^3}{3a^2} - b^2x + C$ which is to vanish when $x = a$, therefore $C = \frac{2}{3}ab^2$ and $W = \frac{b^2x^3}{3a^2} - b^2x + \frac{2}{3}ab^2 = \frac{b^2x^3}{3a^2} - \frac{1}{3}b^2x - \frac{2}{3}b^2x + \frac{2}{3}ab^2 = \frac{1}{3}y^2x - \frac{2}{3}b^2 \times AO$ therefore $g = \frac{M}{W} = \frac{3a^2y^4}{4b^2y^2x - 8b^4 \times AO}$.

Example

Example IX. *Fig.* 84.

303. *To find the center of gravity of the surface described by the revolution of the circular arc* AP *round the axis* AC.

Call AC, r; AO, x; OP, y; let A be the point of suspension, then by art. 285. $W = 2rx$, $\dot{M} = x\dot{W} = 2rx\dot{x}$, whence $M = rx^2$ and $Cg = \frac{M}{W} = \frac{1}{2}x = \frac{1}{2}AO$.

Lemma X.

304. *Let* G *(fig.* 86.*) be the center of gravity of a system of bodies* A, B, C, *and to any plane* aS *draw the perpendiculars* Aa, Bb, Cc, Gg, *then* $Gg \times \overline{A + B + C}$ *is equal to* $A \times Aa + B \times Bb + C \times Cc$.

Take any point S in the plane aS beyond the point C and draw $S\kappa$ perpendicular to aS and $A\alpha$, $B\beta$, $G\gamma$, $C\kappa$, perpendicular to $S\kappa$, then by art. 294. $S\gamma = \frac{M}{W}$ and $S\gamma \times W = M = S\alpha \times A + S\beta \times B + S\kappa \times C$, or $Gg \times W = Aa \times A + Bb \times B + Cc \times C$. Q. E. D.

Of the Center of Oscillation.

305. Let SABC (fig. 87.) be a compound pendulum composed of several bodies A, B, C, vibrating about the axis of motion at S, and if SV is the length of a simple pendulum, which performs its vibrations in the same time, and with the same angular velocity as the compound pendulum. Then the point V is called the center of Oscillation of the system of bodies A, B, C.

Prop. L.

306. *To find the center of oscillation of any system of bodies* A, B, C. (fig. 87.)

Suppose the bodies A, B, C to be in the same plane parallel to the horizon, a section of which is represented by the line SG. Let G be the center of Gravity and V the center of Oscillation. Let all the points A, B, C, G, V, by their gravity descend to a, b, c, g, v. Then by Lemma X. $Gg \times W = aA \times A + bB \times B + cC \times C$. But the spaces described by falling bodies are as the squares of the velocities, that is, as the squares of the distances from S, therefore $SV^2 : SA^2 :: Vv : Aa$ and $Aa = Vv \times \frac{SA^2}{SV^2}$, in the same manner $Bb = Vv \times \frac{SB^2}{SV^2}$ and $Cc = Vv \times \frac{SC^2}{SV^2}$.

Therefore $Gg \times W = Vv \times \frac{SA^2 \times A + SB^2 \times B + SC^2 \times C}{SV^2}$ and $\frac{SV^2 \times Gg \times W}{Vv}$ $= SA^2 \times A + SB^2 \times B + SC^2 \times C$.

But

But SV : Vv :: SG : Gg since the point V has the same angular velocity as the point G, therefore $\frac{SV \times Gg}{Vv} = SG$, and $SV \times SG \times W = SA^2 \times A + SB^2 \times B + SC^2 \times C$, whence $SV = \frac{SA^2 \times A + SB^2 \times B + SC^2 \times C}{SG \times W}$. Therefore the distance of the center of Oscillation from the axis of motion is found by dividing the sum of the products of the squares of the distances of each body into its weight, by the distance of the center of gravity from the same axis of motion multiplied by the sum of the bodies.

Corol. 1. Call the sum of the products $SA^2 \times A$, $SB^2 \times B$, $SC^2 \times C$, P; then $SV = \frac{P}{SG \times W} = \frac{P}{gW} = \frac{P}{M}$; if the distance from $S = x$ then $\dot{M} = x\dot{W}$, and $\dot{P} = x^2\dot{W} = x\dot{M}$.

Corol. 2. Call the sum of all the products of each particle multiplied by the square of its distance from the center of gravity D, then $\dot{D} = \overline{x-g}|^2 \times \dot{W} = x^2\dot{W} - 2gxW + g^2\dot{W} = \dot{P} - 2g\dot{M} + g^2\dot{W}$ and $D = P - 2gM + g^2W = P - gM$ therefore $\frac{D}{M} + g = \frac{P}{M} = SV$, and GV the distance of the center of oscillation from the center of gravity is equal to $\frac{D}{M} = \frac{D}{gW}$.

Corol. 3. If the axis of motion be altered from S to s (fig. 93.) let the center of oscillation be changed from V to v, put $Ss = d$, then $GV = \frac{D}{gW}$, and $Gv = \frac{D}{\overline{g-d} \times W}$. Therefore $GV : Gv :: g-d : g :: sG : SG$, therefore the distance of the center of oscillation from the center of gravity is reciprocally as the distance of the center of gravity from the axis of motion; the position of the axis of motion with respect to the body remaining the same.

Corol. 4. To find the center of oscillation of the area AOP (fig. 82.) let A be the beginning of the abscissa and G the center of gravity; call SA, d; AO, x; GO, z; OP, y; then $\dot{P} = \overline{d+x}^2 \times y\dot{x}$, and $\dot{D} = z^2 y\dot{x}$. and $SV = g + \frac{D}{gW} = \frac{P}{gW}$.

Lemma XI.

307. *Let* AOPQ *(fig. 94.) be a solid generated by the revolution of the area* AOP *about the axis* AO. *To find the sum of all the products of each particle of the solid multiplied by the square of its distance from a given point* S.

Let the circle Pn QMN be a section of the solid by a plane perpendicular to AO, let nRN be perpendicular to OP. Call SA, d; AO, x; OP, y; OR, v; the area ORN, A; and put $v^2\dot{A} = \dot{B}$, then the fluxion of the sum of the four areas ORN, ORn, Orm, OrM will be $4\dot{A}$, which multiplied by

by the square of the distance from S, or $\overline{d+x}|^2 +$ v^2 gives $4 \times \overline{d+x}|^2 \times \dot{A} + 4v^2\dot{A} = 4\dot{A} \times \overline{d+x}|^2$ $+ 4\dot{B}$ whose fluent is $\overline{d+x}|^2 \times 4A + 4B$; but by art. 200. the whole fluent $B = \frac{1}{4}Ay^2$ therefore the fluent is $\overline{d+x}|^2 \times 4A + Ay^2$, where A is the quarter of a circle whose radius is y, and the whole area $4A = py^2$, (p being the area of a circle whose radius is 1) therefore the fluent is $\overline{d+x}|^2 \times$ $py^2 + \frac{1}{4}py^4$ which is the sum of all the particles in the circle PQ, multipled by the squares of their distances from S. Therefore multiply this last fluent by $\dot{x}$ and find its fluent which will be the sum of all the particles in the solid, multiplied by the squares of their respective distances from S. Q. E. I.

Corol. 1. To find the center of oscillation of a solid then $\dot{P} = py^2\dot{x} \times \overline{d+x}|^2 + \frac{1}{4}py^4\dot{x}$.

If the point of suspension is at A then $\dot{P} =$ $py^2x^2\dot{x} + \frac{1}{4}py^4\dot{x}$.

Corol. 2. If G is the center of gravity and GO $= z$, then $\dot{D} = py^2z^2\dot{z} + \frac{1}{4}py^4\dot{z}$.

EXAMPLES *of finding the Center of Oscillation.*

Example I. *Fig.* 88.

308. *To find the Center of Oscillation of a right line* SB.

Call SB, a; SO, x; then $\dot{P}=x^2\dot{x}$ and $P=\frac{1}{3}x^3$, and when $x=a$, $P=\frac{1}{3}a^3$, but $W=a$ and $SG=\frac{1}{2}a$, therefore $SV=\frac{\frac{1}{3}a^3}{\frac{1}{2}a^2}=\frac{2}{3}a=\frac{2}{3}SB$.

Example II. *Fig.* 89.

309. *To find the Center of Oscillation of a circle.*

Let G be the center of the circle, and let the axis of motion be at S; call SG, g; AG, r; GO, z; OP, y; and supposing the circle ABC to be divided into an infinite number of concentrical circles OLM, then $\dot{D}=z^2\dot{z}\times OLM=2pz^3\dot{z}$, and $D=\frac{1}{2}pz^4$ and when $z=r$, $D=\frac{1}{2}pr^4$. But $W=pr^2$, therefore $GV=\frac{D}{SG\times W}=\frac{r^2}{2\times SG}=\frac{GA^2}{2\times SG}$.

Example

Example III. *Fig.* 82.

310. *To find the Center of Oscillation of a parabola whose equation is $y = ax^m$.*

Let the axis of motion be at S, call SA, d; AO, x; OP, y; then $\dot{P} = \overline{d+x}^2 \times a\dot{x}x^m = ad^2\dot{x}x^m + 2ad\dot{x}x^{m+1} + a\dot{x}x^{m+2}$, and $P = \frac{ad^2x^{m+1}}{m+1} + \frac{2adx^{m+2}}{m+2} + \frac{ax^{m+3}}{m+3}$. But $W = \frac{ax^{m+1}}{m+1}$ (art. 204.) therefore $SV = \frac{P}{gW} = \frac{d^2}{g} + \frac{2.\overline{m+1}.dx}{\overline{m+2}.g} + \frac{\overline{m+1}.x^2}{\overline{m+3}.g}$.

If the axis of motion is at A then $d = 0$, and $AV = \frac{m+1}{m+3} \times \frac{x^2}{g}$. But $g = \frac{m+1}{m+2} \times x$ therefore $AV = \frac{m+2}{m+3} \times x$.

In a triangle $m = 1$, and $AV = \frac{3}{4}AO$.

In the common parabola, $m = \frac{1}{2}$, and $AV = \frac{5}{7}$ AO.

In the cubical parabola, $m = \frac{1}{3}$, and $AV = \frac{7}{10}$ AO.

Example IV. *Fig.* 89.

311. *To find the center of Oscillation of a sphere.*

Supposing the same things as in art. 309. let the sphere be supposed to be divided into an infinite number

number of cylindric surfaces, which are terminated at the surface of the sphere, and whose bases are the circles OLM, then each of those surfaces will be equal $2p \times GO \times OP = 2pzy$ and $\dot{D} = 2pyz^3\dot{z} = 2py^2z^2\dot{y} = 2pr^2y^2\dot{y} - 2py^4\dot{y}$ and $D = \frac{2}{3}pr^2y^3 - \frac{2}{5}py^5$ which vanishes when $y=0$, and when $y=r$, $D = \frac{4}{15}pr^5$. But $W = \frac{2}{3}pr^3$ therefore $GV = \frac{D}{gW} = \frac{2r^2}{5g} = \frac{2AG^2}{5SG}$.

312. The center of Oscillation of a sphere may also be found thus. By Lemma XI. Cor. 2. we have $\dot{D} = py^2z^2\dot{z} + \frac{1}{4}py^4\dot{z}$. But $y^2 = r^2 - z^2$ and $\frac{1}{4}y^4 = \frac{1}{4}r^4 - \frac{1}{2}r^2z^2 + \frac{1}{4}z^4$ therefore $\dot{D} = \frac{1}{4}pr^4\dot{z} + \frac{1}{2}pr^2z^2\dot{z} - \frac{3}{4}pz^4\dot{z}$ and $D = \frac{1}{4}pr^4z + \frac{1}{6}pr^2z^3 - \frac{3}{20}pz^5$, and when $z=r$, $D = \frac{4}{15}pr^5$ as before.

Example V. *Fig.* 82.

313. *To find the center of Oscillation in the solid generated by the revolution of a parabola, whose equation is $y = ax^m$ round the axis* AO.

Let the axis of motion be at A. Call AO, x; OP, y; then by Lemma XI. $\dot{P} = py^2x^2\dot{x} + \frac{1}{4}py^4\dot{x} = pa^2\dot{x}x^{2m+2} + \frac{1}{4}pa^4\dot{x}x^{4m}$, and $P = \frac{pa^2x^{2m+3}}{2m+3} + \frac{pa^4x^{4m+1}}{4 \times 4m+1}$. But by art. 300. $M = \frac{pa^2x^{2m+2}}{2m+2}$ therefore $SV = \frac{P}{M} = \frac{2m+2}{2m+3} \times x + \frac{m+1}{2.4m+1} \times$

a^2

$$a^2x^{2m-1} = \frac{2m+2}{2m+3} \times x + \frac{m+1}{2 \cdot 4m+1} \times \frac{y^2}{x}.$$

In a cone, $m = 1$, and $SV = \frac{4}{5} AO + \frac{OP^2}{5AO}$.

In a parabolic conoid, $m = \frac{1}{2}$, and $SV = \frac{3}{4} AO + \frac{OP^2}{4AO}$.

PROP. LI.

314. *Let* AOP *(fig. 95.) be a curve,* AO *the abscissa, to find the motion of the solid generated by* AOP *when it revolves round the axis* AO.

Let OP be an ordinate, R any point in it, let the points R, r, be infinitely near each other; call AO, x; OP, y; OR, z, then the motion of the annulus described by Rr will be as that annulus and its velocity conjointly, that is, as OR $\times$ annulus Rr $= 2p \times OR^2 \times Rr$, and therefore the fluxion of the motion of the circle described by OP is equal to $2pz^2\dot{z}$ whose fluent is $\frac{2}{3}pz^3$ and when $z = y$, the fluent is $\frac{2}{3}py^3$, which is the motion of the circle described by OP, therefore the fluxion of the motion of the solid described by AOP is as $\frac{2}{3}py^3\dot{x}$, that is, as $y^3\dot{x}$. The fluent of which will be as the motion of AOP. Q. E. I.

Corol. If AL $= a$, then the motion of the cylinder ALKO will be to the motion of the solid AOP as a^3x to $\overline{|y^3\dot{x}|}$.

Example

Example. Fig. 95.

315. *To find the motion of a sphere revolving round the axis* AC.

Call the radius AC, r; CO, x; OP, y; then $y^3\dot{x} = \overline{r^2 - x^2} \times y\dot{x}$ put the fluxion of the area COPD $= y\dot{x} = \dot{A}$, and $\dot{A}x^2 = \dot{B}$ then the fluxion of the motion $y^3\dot{x} = r^2\dot{A} - \dot{B}$ whose fluent is $r^2A - B$, but by art. 200, the fluent B when $x = r$ is equal to $\frac{1}{4}r^2A$, therefore the motion of the spherical segment ACD $= \frac{3}{4}r^2A$, and the motion of the whole sphere $= \frac{3}{2}r^2A$, and the motion of the sphere is to the motion of the circumscribed cylinder ALDC as $r^4 : \frac{3}{4}r^2A :: 4r^2 : 3A$, that is, as four circumscribed squares is to three great circles.

Prop. LII.

316. *Let* BAK *(fig. 90.) be a perfectly flexible line hanging from the two points* B,K, *and by its gravity formed into the curve* BAK, *to find the figure of this curve.*

Let A be the lowest point, through which draw AO perpendicular to the horizon, and from any point P, draw the line OP perpendicular to AO, call AO, x; OP, y; the arc AP, t; and let the tension of the line at A be equal to the weight of a line whose length is a, let PT be a tangent at the point P, then the line at P is acted upon by three forces, by its gravity in a line parallel to AO, by the

the arc BP in the direction of the tangent TP, and by the tension of AK at A in a direction parallel to OP. But it appears by mechanics, that since the line AP is in *æquilibrio* these forces must be to each other as the sides of a triangle parallel to the directions of those forces, therefore the tension at A is to the weight of the arc AP as OP to OT, that is, $a : t :: \dot{y} : \dot{x}$ whence $a^2 : a^2 + t^2 :: \dot{y}^2 : \dot{y}^2 + \dot{x}^2 :: \dot{y}^2 : \dot{t}^2$ and $\dot{y} = \frac{a\dot{t}}{\sqrt{a^2 + t^2}}$ which fluxion may be compared with that in art. 94. Exam. 3. where $d = a$, $\eta = 2$, $e = a^2$, $f = 1$, $R = 1$, $T = \frac{1}{t}\sqrt{a^2 + t^2}$, $S = \frac{a}{t}$, and the fluent $\frac{2}{\eta f} \times dR \left| \frac{R + T}{S} \right. = a \left| \frac{t + \sqrt{a^2 + t^2}}{a} \right. = y$. In the right line OA produced, take $AF = a$; let AE be perpendicular to AO at A, take $AE = AP$ then $OP = AF \left| \frac{AE + EF}{AF} \right.$.

Since $\dot{x} = \frac{t\dot{y}}{a}$ and $\dot{y} = \frac{a\dot{t}}{\sqrt{a^2 + t^2}}$ we have $\dot{x} = \frac{t\dot{t}}{\sqrt{a^2 + t^2}}$. Put $a^2 + t^2 = z^2$ then $t\dot{t} = z\dot{z}$, and $\dot{x} = \frac{t\dot{t}}{z} = \dot{z}$, therefore $z = x + C$ which is to vanish when $t = 0$, or $z = a$, therefore $C = a$, and $a + x = z$, or $OF = FE$.

Definition. This curve is called the *Catenaria*.

Corol.

Corol. 1. Since $\dot{x} = \frac{t\dot{y}}{a}$ therefore the subtangent $OT = \frac{y\dot{x}}{\dot{y}} = \frac{ty}{a} = \frac{AE \times OP}{AF}$, the subnormal $OD = \frac{ay}{t} = \frac{AF \times OP}{AE}$, and the perpendicular $DP = \frac{y}{t}\sqrt{a^2+t^2} = \frac{OP \times FE}{AE}$.

Corol. 2. Put $\frac{\dot{y}}{\dot{x}} = z$ then $z = \frac{a}{t}$ and $\dot{z} = -\frac{a\dot{t}}{t^2}$, but $\dot{x} = \frac{t\dot{t}}{\sqrt{a^2+t^2}}$ therefore $\frac{\dot{x}}{\dot{z}} = \frac{t^3}{a\sqrt{a^2+t^2}} = \frac{AE^3}{AF \times FE}$. Also $\frac{DP}{OP} = \frac{FE}{AE}$ therefore the radius of curvature in P is equal $\frac{DP^3}{OP^3} \times \frac{\dot{x}}{\dot{z}}$ (art. 152.) $= \frac{FE^3}{AE^3} \times \frac{AE^3}{AF \times FE} = \frac{FE^2}{AF}$.

SECT. XIII.

Of CENTRIPETAL FORCES.

PROP. LIII.

317. *If a body falls in a right line towards the center of force, the increment of the velocity will be as the force that generates it, and as the time in which it is generated conjointly.*

For the greater the force is and the longer it acts, the greater effect it must produce, that is, it must produce a greater change of velocity, which is therefore as the force and time conjointly. Q. E. D.

Corol. 1. Call the force F, the time t, the velocity v, the distance from the center y, then $\dot{v} = F\dot{t}$ and $F = \frac{\dot{v}}{\dot{t}}$.

Corol. 2. Since the space is as the time and velocity conjointly $\dot{y} = v\dot{t}$, and $\dot{t} = \frac{\dot{y}}{v}$, therefore $F = \frac{v\dot{v}}{\dot{y}}$ whence $\dot{y}F = v\dot{v}$ and taking the fluents $v^2 = 2 \times \left|\underline{\dot{y}F}\right|$ or since 2 is given, v^2 is as $\left|\underline{\dot{y}F}\right|$ *.

* Newton. Princip. Lib. I. Prop. 39.

PROP.

Prop. LIV.

318. *If a body moves in a curve line, the force towards the center will be to the force that accelerates the velocity as the fluxion of the curve to the fluxion of the distance from the center of force.*

Let the center of force be S (fig. 96.) EP the curve the body moves in, let the two points P, *p*, be infinitely near each other, let fall the perpendicular *pn* on SP and suppose the force towards S to be represented by P*n*, which may be resolved into two forces PT, and T*n*, one in the direction P*p*, and the other perpendicular to it, of which PT will only accelerate the body in the curve, and the force P*n* : force PT :: P*n* : PT :: P*p* : P*n*. Q. E. D.

Corol. 1. Call the arc EP, *s*; the accelerating force A, then F : A :: $\dot{s}$: $\dot{y}$ and $F = \frac{A\dot{s}}{\dot{y}}$. But by Prop. LIII, $A = \frac{\dot{v}}{\dot{t}}$, therefore $F = \frac{\dot{v}\dot{s}}{\dot{t}\dot{y}}$. or since $\frac{\dot{s}}{\dot{t}} = v$, $F = \frac{v\dot{v}}{\dot{y}}$, therefore (Prop. LIII. Cor. 2.) the velocity is the same in a right line and in a curve at the same distances from the center of force, if the forces at those distances are equal.

Corol. 2. If $\dot{t}$ be given then v is as $\dot{s}$, $\dot{v} = \ddot{s}$, and $F = \frac{\dot{s}\ddot{s}}{\dot{y}}$.

Corol.

Corol. 3. Call the force Tn that acts perpendicular to the curve, N : then $N : F :: nT : Pn :: pn : Pp$, let SY (fig. 97.) be perpendicular to the tangent PY, call SY, z; then $pn : Pp :: SY : SP$ and $N = \frac{Fz}{y} = \frac{zv\dot{v}}{y\dot{y}}$:

Corol. 4. Let SA be the height a body must fall to acquire the velocity it has in D or P, call SA, c, and let the centripetal force be as the power $n-1$ of the distance, then $F\dot{y} = \dot{y}y^{n-1} =$ (cor. 1.) $v\dot{v}$ and $v^2 = \frac{2}{n} \times y^n + C$, that is, as $y^n + C$ or as $C - y^n$ which is to vanish when $y = c$, therefore $C = c^n$, and v^2 is as $c^n - y^n$ and v as $\sqrt{c^n - y^n}$.

Prop. LV.

319. *If a body moves in a curve line, the force will be as the square of the velocity directly, and as the chord of the circle of curvature passing through the center of force inversely.*

Suppose the time in which the body describes the infinitely small arc Pp (fig. 97.) to be given, then, if the force was to cease acting in P, the body by its *vis insita* would go on in the tangent Pr, but if the force acts it is obliged to move in the curve Pp, which is the diagonal of a parallelogram whose sides are pr, Pr, therefore the force is as pr, but if PV is the chord of curvature, $PV \times pr = Pp^2$ therefore $pr = \frac{Pp^2}{PV}$. But Pp is as the velocity because the time is given, therefore the force is as

S the

the ſquare of the velocity directly and PV inverſely. Q. E. D.

Corol. 1. Call PV, w; SY, z; then by art. 177, $w = \frac{2z\dot{y}}{\dot{z}}$, and $F = \frac{v^2}{w} = \frac{v^2\dot{z}}{2z\dot{y}}$. But by Prop. LIV. Cor. 1. $F = \frac{-v\dot{v}}{\dot{y}}$, therefore ſince 2 is given $\frac{v^2\dot{z}}{z\dot{y}} = \frac{-v\dot{v}}{\dot{y}}$ and $v\dot{z} + \dot{v}z = 0$ and taking the fluents vz is given and v inverſely as z.

Corol. 2. Therefore $\dot{v} = \frac{-\dot{z}}{z^2}$, and $F = \frac{-v\dot{v}}{\dot{y}} = \frac{\dot{z}}{z^3\dot{y}}$. and N (Prop. LIV. Corol. 3.) $= \frac{Fz}{y} = \frac{\dot{z}}{z^2 y\dot{y}}$. Let R be the radius of curvature then $R = \frac{y\dot{y}}{\dot{z}}$, and $F = \frac{\dot{z}y}{z^3 y\dot{y}} = \frac{y}{z^3 R}$, and $N = \frac{1}{z^2 R} = \frac{v^2}{R}$.

Corol. 3. Since $\dot{t} = \frac{\dot{s}}{v}$ and $v = \frac{1}{z}$ we have $\dot{t} = z\dot{s} = SY \times Pp$, that is, as the area SPp.

Corol 4. Since $F = \frac{v^2}{w}$ we have $Fw = v^2$ and $w = \frac{v^2}{F} = \frac{v^2\dot{y}}{v\dot{v}} = \frac{v\dot{y}}{\dot{v}}$.

Corol.

Corol. 5. Since $\frac{\dot z}{z^3} = F\dot y$ we have $\frac{1}{z^2} = \boxed{2F\dot y}$ and $F = \frac{\dot z}{z^3\dot y} = \frac{\dot z}{z\dot y} \times \boxed{2F\dot y} = \frac{2}{w} \times \boxed{2F\dot y}$ and $\frac{1}{4}Fw = \boxed{F\dot y}$.

Corol. 6. Call the velocity in a circle at the same distance, V; then $V^2 : v^2 :: 2Fy : Fw :: \frac{1}{2}Fy : \boxed{F\dot y}$. Therefore the velocity in a circle whose radius is y, is as $\sqrt{\frac{1}{2}Fy}$, or as $\sqrt{Fy}$ and $F = \frac{V^2}{y}$.

Corol. 7. Let E be the area described in a given time 2, then by Cor. 3; $2 : E :: \dot t : \frac{1}{2}z\dot s$ and $E = \frac{z\dot s}{\dot t} = zv = z\sqrt{Fw}$ therefore $E^2 = z^2Fw$ and $F = \frac{E^2}{z^2w}$. In a circle whose radius is y, $z = y$, and $w = 2y$ therefore the force in a circle is as $\frac{E^2}{2y^3}$ or as $\frac{E^2}{y^3}$.

Corol 8. Call the arc BF, x, SB, r; then $z = \frac{y^2\dot x}{r\dot s}$ and $\dot t = z\dot s = \frac{y^2\dot x}{r}$ and $\dot x = \frac{r\dot t}{y^2}$, therefore the angular velocity which is as $\frac{\dot x}{\dot t}$ is inversely as the square of the distance.

Corol. 9. Since $\frac{\dot s}{v} = \dot t = \frac{y^2\dot x}{r}$ we have $\frac{\dot s^2}{v^2} = \frac{y^4\dot x^2}{r^2}$ or $\frac{\dot y^2r^2 + y^2\dot x^2}{v^2} = y^4\dot x^2$.

Prop. LVI.

Having the orbit to find the centripetal force.

Compute the quantity $\frac{1}{z^2 w}$ or $\frac{\dot{z}}{z^3 \dot{y}}$ and it will be as the force.

Example I. *Fig.* 6.

320. *To find the force that tends to the center of an ellipse.*

Call SN, p; AS, a; SG, b; SP, y; then $w = \frac{2p^2}{y}$ and $zp = ab$, therefore $z^2 = \frac{a^2 b^2}{p^2}$ and $\frac{1}{z^2 w} = \frac{y}{2a^2 b^2}$, therefore the force is directly as the distance.

Example II.

321. *If a body moves in an ellipse, to find the force that tends to the focus of that ellipse.*

Let the tranverse axis $= 2a$, the conjugate $= 2b$ then the distance from the other focus $= 2a - y$, and by art. 127. $z = \frac{by}{\sqrt{2ay - yy}}$ and $\frac{1}{z^2} = \frac{2ay - yy}{b^2 y^2} = \frac{2a}{b^2 y} - \frac{1}{b^2}$, therefore $\frac{\dot{z}}{z^3 \dot{y}} = \frac{a}{b^2 y^2}$, that is, the force is inversely as the square of the distance.

Example

Example III. *Fig.* 21.

322. *To find the force tending to the center of a spiral.*

Call SP, y; SB, r; BF, x; SY, z; and let $x : r :: y^m : a^m$ then by art. 122. $z^2 = \frac{m^2 y^{2m+2}}{a^{2m} + m^2 y^{2m}}$ and $\frac{1}{z^2} = \frac{a^{2m}}{m^2 y^{2m+2}} + \frac{1}{y^2}$ therefore $\frac{\dot{z}}{z^3 \dot{y}} = \frac{\overline{m+1} . a^{2m}}{m^2 y^{2m+3}} + \frac{1}{y^3}$ which is as the centripetal force tending to S.

If $m = 1$, the force that tends to the center of the spiral of Archimedes is as $\frac{2a^2}{y^5} + \frac{1}{y^3}$.

If $m = -1$, the force tending to the center of the reciprocal spiral is as $\frac{1}{y^3}$, that is, inversely as the cube of the distance.

If $m = -\frac{1}{2}$, that is, if y is inversely as x^2, the force is as $\frac{2}{ay^2} + \frac{1}{y^3}$.

If $\overline{m+1}$ is negative, then the first part $\frac{\overline{m+1} . a^{2m}}{m^2 y^{2m+3}}$ will be negative, and the force will be part affirmative and part negative, or part centripetal and part centrifugal, because the curve has a point of contrary flexure. Art. 190.

Example IV.

323. *To find the force tending the center of an equiangular spiral.*

Since z is to y in a given ratio as c to r we have $z = \frac{cy}{r}$ and $\frac{1}{z^2} = \frac{r^2}{c^2y^2}$ whence the force $\frac{\dot{z}}{z^3\dot{y}} = \frac{r^2}{c^2y^3}$, that is, inversely as the cube of the distance.

Example V. *Fig.* 98.

324. *Let* VR *be any curve, let the curve* VP *be formed by taking* SP = SR, *and the angle* VSR *to the angle* VSP *in a given ratio as* F *to* G, *to find the difference between the forces tending to* S, *in the trajectories* VP, VR.

Call SP = SR, y; SB, r; BE, x; BF, X, let SG, SY, be perpendicular to the tangents RG, PY, call SG, z; SY, Z; the arc VR, s, then $\dot{s}^2 = \dot{y}^2 + \frac{y^2\dot{x}^2}{r^2}$ and $\frac{1}{z^2} = \frac{r^2\dot{s}^2}{y^4\dot{x}^2} = \frac{r^2\dot{y}^2}{y^4\dot{x}^2} + \frac{1}{y^2}$, in the same manner $\frac{1}{Z^2} = \frac{r^2\dot{y}^2}{y^4\dot{X}^2} + \frac{1}{y^2} = \frac{F^2r^2\dot{y}^2}{G^2y^4\dot{x}^2} + \frac{1}{y^2}$ (because G : F :: X : x) but $\frac{r^2\dot{y}^2}{y^4\dot{x}^2} = \frac{1}{z^2} - \frac{1}{y^2}$ therefore $\frac{1}{Z^2} = \frac{F^2}{G^2z^2} -$

F²

$\frac{F^2}{G^2y^2} + \frac{1}{y^2}$, put $G^2 - F^2 = P$ then $\frac{1}{Z^2} =$ $\frac{F^2}{G^2z^2} + \frac{P}{G^2y^2}$, and taking the fluxions $\frac{\dot{Z}}{Z^3\dot{y}} =$ $\frac{F^2\dot{z}}{G^2z^3\dot{y}} + \frac{P}{G^2y^3}$. But $\frac{\dot{Z}}{Z^3\dot{y}}$ is as the force in the curve VP and $\frac{F^2\dot{z}}{G^2z^3\dot{y}}$ is as the force in the curve VR and the difference of the forces $\frac{P}{G^2y^3}$ is inversely as the cube of the distance *.

Cor. 1. If the radius of curvature in R be called ϱ, then $\varrho = \frac{y\dot{y}}{\dot{z}}$, and $\frac{\dot{z}}{\dot{y}} = \frac{y}{\varrho}$ and the force in VP will be as $\frac{F^2y}{G^2z^3\varrho} + \frac{P}{G^2y^3}$, or if N is a given quantity the force is as $\frac{NF^2y}{G^2z^3\varrho} + \frac{NP}{G^2y^3}$.

Corol. 2. Call SV the greatest distance of the body from S, T; the radius of curvature in V, R; and let the centripetal force in the curve VR in V, be $\frac{VF^2}{T^2}$: then $\frac{NF^2y}{G^2z^3\varrho}$ or $\frac{NF^2}{G^2T^2R} = \frac{VF^2}{T^2}$ and $N =$ VRG^2 and the force in $P = \frac{F^2VRy}{z^3\varrho} + \frac{VRP}{y^3}$. If we call the force in R, W; then the force at $P =$ $W + \frac{VRP}{y^3}$.

If VRK is an ellipse whose focus is S, then the

* Newton Princip. lib. i. prop. 44.

force in R is as $\frac{F^2}{y^2}$, $V=1$, and the force in $P =$ $\frac{F^2}{y^2} + \frac{RP}{y^3}$.

Corol. 3. Call the radius of curvature at P, d, then $\frac{\dot{Z}}{\dot{y}} = \frac{y}{d}$ and $\frac{y}{Z^3 d} = \frac{F^2 y}{G^2 z^3 \varrho} + \frac{P}{G^2 y^3}$ or $\frac{1}{Z^3 d} = \frac{F^2}{G^2 z^3 \varrho} + \frac{P}{G^2 y^4}$.

Call the curvature of the curve VP at P, c; the curvature of VR at R, $\varkappa$, and the curvature of a circle whose radius is y, k; then c, $\varkappa$, k, are as $\frac{1}{d}$, $\frac{1}{\varrho}$, $\frac{1}{y}$ therefore $\frac{c}{Z^3} = \frac{F^2 \varkappa}{G^2 z^3} + \frac{Pk}{G^2 y^3} =$ $\frac{F^2 \varkappa}{G^2 z^3} + \frac{k}{y^3} - \frac{F^2 k}{G^2 y^3}$. In the greatest and least distances from S, we have $y = z = Z$ and $c =$ $\frac{F^2 \varkappa}{G^2} + k - \frac{F^2 k}{G^2}$, and $c - k = \frac{F^2}{G^2} \times \overline{\varkappa - k}$, therefore $c - k : \varkappa - k :: F^2 : G^2$.

PROP.

Prop. LVII.

325. *Having the force to find the orbit.*

By Prop. LV. Corol. 9. we have $\frac{\dot{y}^2r^2+y^2\dot{x}^2}{y^4\dot{x}^2}$ as v^2, or if a is a given quantity $\frac{\dot{y}^2r^2+y^2\dot{x}^2}{y^4\dot{x}^2} = \frac{v^2}{a^2} = \frac{1}{a^2} \times \boxed{F\dot{y}}$.

If the force is as y^{n-1} and c is the distance from which the body must fall to acquire the velocity it has at the distance y then by Prop. LIV. Corol. 4. $v^2 = c^n - y^n$ therefore $\frac{\dot{y}^2r^2+y^2\dot{x}^2}{y^4\dot{x}^2} = \frac{c^n-y^n}{a^2}$ and $a^2r^2\dot{y}^2 + a^2y^2\dot{x}^2 = c^ny^4\dot{x}^2 - y^{n+4}\dot{x}^2$, whence $\dot{x}^2 = \frac{a^2r^2\dot{y}^2}{c^ny^4-y^{n+4}-a^2y^2}$ and $\dot{x} = \frac{ar\dot{y}}{y\sqrt{c^ny^2-y^{n+2}-a^2}}$ which is the equation of the curve.

Corol. 1. In the apsides $\dot{y} = 0 = \frac{\dot{x}y}{ar} \times \sqrt{c^ny^2-y^{n+2}-a^2}$, therefore the equation to determine the apsids is $c^ny^2-y^{n+2}-a^2=0$ or $y^{n+2}-c_ny^2+a^2=0$.

Corol. 2. Hence if n is a whole and affirmative number, the number of apsids can never exceed $n+2$, if $n+2$ is an integral negative number as $-m$, the number of apsids will never exceed $m+2$.

Of

Of the Resistance of Bodies.

Prop. LVIII.

326. *If a body moves forward by its* vis insita *only the resistance is as the decrement of the velocity directly and as the time inversely.*

For the decrement of the velocity is as the resistance that destroys it and the time conjointly, therefore the resistance is as the decrement of the velocity directly and time inversely.

Corol. 1. Call the velocity v, the time t, the space s, resistance R, then $R = \frac{\dot{v}}{\dot{t}}$ and $\dot{t} = \frac{\dot{v}}{R}$ and $t = \left|\frac{\dot{v}}{R}\right|$. If the resistance is as the power n of the velocity, let $R = av^n$ then $\dot{t} = \frac{\dot{v}}{av^n}$ and $t = \frac{v^{1-n}}{\overline{1-n}\,.\,a}$, that is, inversely as v^{n-1}. if $n = 1$, then $\dot{t} = \frac{\dot{v}}{av}$ and $t = \frac{1}{a} \times \log.\ v$.

Corol. 2. Since $\dot{s} = v\dot{t}$ and $\dot{t} = \frac{\dot{v}}{R}$ we have $\dot{s} = \frac{v\dot{v}}{R}$, and $s = \left|\frac{v\dot{v}}{R}\right|$. If $R = av^n$, $\dot{s} = \frac{v\dot{v}}{av^n} = \frac{\dot{v}v^{1-n}}{a}$ and $s = \frac{v^{2-n}}{a\,.\,\overline{2-n}}$. If $n = 2$, $\dot{s} = \frac{\dot{v}}{av}$ and $s = \frac{1}{a} \times \log.\ v$.

Prop.

Prop. LIX.

327. If a body moves in a right line, and is acted upon by a gravity F, and a resistance R, then if the body descends, the decrement of the velocity will be as the difference of the gravity and resistance, and as the time conjointly, but if it ascends it will be as the sum of F and R and time conjointly.

For when the body descends the accelerating force is F — R and when it ascends the retarding force is F+R, therefore by Prop. LIII. $F \pm R = \frac{\dot{v}}{\dot{t}}$. Q. E. D.

Corol. 1. Hence $\dot{t} = \frac{\dot{v}}{F \pm R}$ and $\dot{s} = v\dot{t} = \frac{v\dot{v}}{F \pm R}$.

Corol. 2. If F is given, and $R = v^n$ then $\dot{t} = \frac{\dot{v}}{F \pm v^n}$ whose fluent may always be expressed by the measures of angles or ratios.

If $n = 0$, $\dot{t} = \frac{\dot{v}}{F \pm 1}$ and $t = \frac{v}{F \pm 1}$.

If $n = 1$, $\dot{t} = \frac{\dot{v}}{F \pm v}$ and $t = \log. \overline{F \pm v}$.

If $n = 2$, and the body descends, then $\dot{t} = \frac{\dot{v}}{F - v^2}$ and $t = \frac{1}{2\sqrt{F}} \times \log. \frac{\sqrt{F} + v}{\sqrt{F} - v}$. But if the body ascends $\dot{t} = \frac{\dot{v}}{F + v^2}$ and $t = \frac{1}{F} \times$ arc whose radius $= \sqrt{F}$ and tangent $= v$.

Corol.

Corol. 3. By Corol. 1, $\dot{s} = \frac{v\dot{v}}{F \pm R} = \frac{v\dot{v}}{F \pm v^n}$.

If $n = 0$, $s = \frac{v^2}{2 \times \overline{F \pm 1}}$.

If $n = 1$, $\dot{s} = \frac{v\dot{v}}{F \pm v} = \pm \dot{v} \mp \frac{F\dot{v}}{F \pm v}$ and $s = \pm v \mp \log. \overline{F \pm v}$.

If $n = 2$, $\dot{s} = \frac{v\dot{v}}{F \pm v^2}$ and $s = \log. \overline{F \pm v^2}$.

Prop. LX.

328. If a body moves in a curve line, call the accelerating force A, then if the body descend towards the center $\frac{\dot{v}}{\dot{t}}$ is as $A - R$, if it ascend $\frac{\dot{v}}{\dot{t}}$ is as $A + R$.

For $A \pm R$ is the force that accelerates or retards the body and by Prop. LIII. $\frac{\dot{v}}{\dot{t}} = A \pm R$.

Corol. 1. Since by Prop. LIV. $A = \frac{F\dot{y}}{\dot{s}}$ in the ascent of the body, and $A = \frac{-F\dot{y}}{\dot{s}}$ in the descent; we have $\pm \frac{\dot{v}}{\dot{t}} = \frac{F\dot{y}}{\dot{s}} + R$ and $F\dot{y} + R\dot{s} = \pm \frac{\dot{v}\dot{s}}{\dot{t}} = \pm v\dot{v}$. If $\dot{t}$ is given then $\dot{v} = \frac{\ddot{s}}{\dot{t}}$ and $\frac{\dot{s}\ddot{s}}{\dot{t}^2} = F\dot{y} + R\dot{s}$.

Corol. 2. Let F be given, then since $F\dot{y} + R\dot{s} = v\dot{v}$ taking the fluents we shall have $Fy + \boxed{R\dot{s}} = \frac{1}{2}v^2$.

Corol.

Corol. 3. If the central force tends to an infinite distance, let y be the ordinate perpendicular to the horizon, x the abſciſſa, and let F and $\dot{x}$ be given, let w be half the chord perpendicular to the abſciſſa, then by Prop. LV. $F = \frac{v^2}{w}$, but by Prop. XXX. $w = \frac{\dot{s}^2}{\ddot{y}}$ therefore $F = \frac{v^2\ddot{y}}{\dot{s}^2}$ and $v^2 = \frac{F\dot{s}^2}{\ddot{y}}$ and taking the fluxions $v\dot{v} = \frac{F\dot{s}\ddot{s}}{\ddot{y}} - \frac{F\dot{s}^2\dddot{y}}{2\ddot{y}^2} = F\dot{y} - \frac{F\dot{s}^2\dddot{y}}{2\ddot{y}^2}$. But $v\dot{v} = F\dot{y} + R\dot{s}$ therefore $R = \frac{F\dot{s}\dddot{y}}{2\ddot{y}^2}$ and $F : R :: 2\ddot{y}^2 : \dot{s}\dddot{y}$.

Corol. 4. Let $R = \frac{v^{2n}}{2a} = \frac{F^n\dot{s}^{2n}}{2a\ddot{y}^n}$ then $\frac{F\dot{s}\dddot{y}}{\ddot{y}^2} = \frac{F^n\dot{s}^{2n}}{a\ddot{y}^n}$ and $F\dot{s}\dddot{y} = \frac{F^n\dot{s}^{2n}}{a\ddot{y}^{n-2}}$ and the equation of the curve is $a\dddot{y} = \frac{F^{n-1}\dot{s}^{2n-1}}{\ddot{y}^{n-2}}$.

If $n = 1$, the equation is $a\dddot{y} = \dot{s}\ddot{y}$.

If $n = \frac{1}{2}$, $a\dddot{y} = F^{-\frac{1}{2}} \times \ddot{y}^{\frac{3}{2}}$.

If $n = 0$, $a\dddot{y} = \frac{\ddot{y}^2}{F\dot{s}}$.

F I N I S.

E R R A T A.

Page 112. line ult. *for* DP^2 *read* DP.

231. l. 3. from bott. and line ult. *for* *c*CO *read* C*eo*.

240. l. penult. *for* AO. *read* CO.

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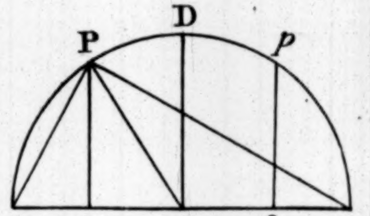

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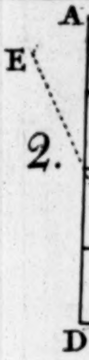

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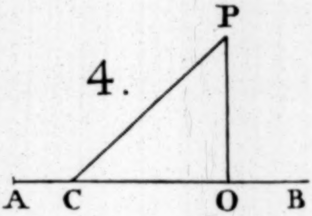

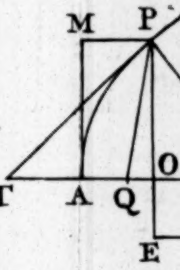

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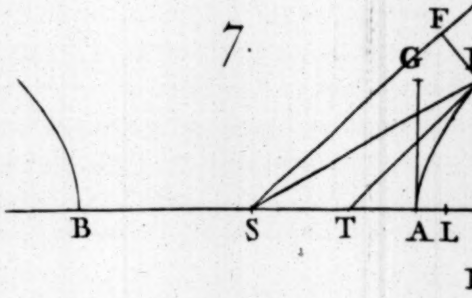

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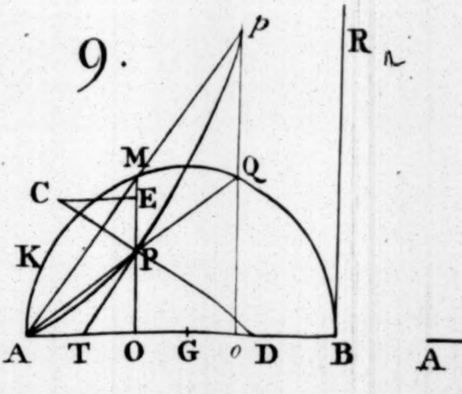

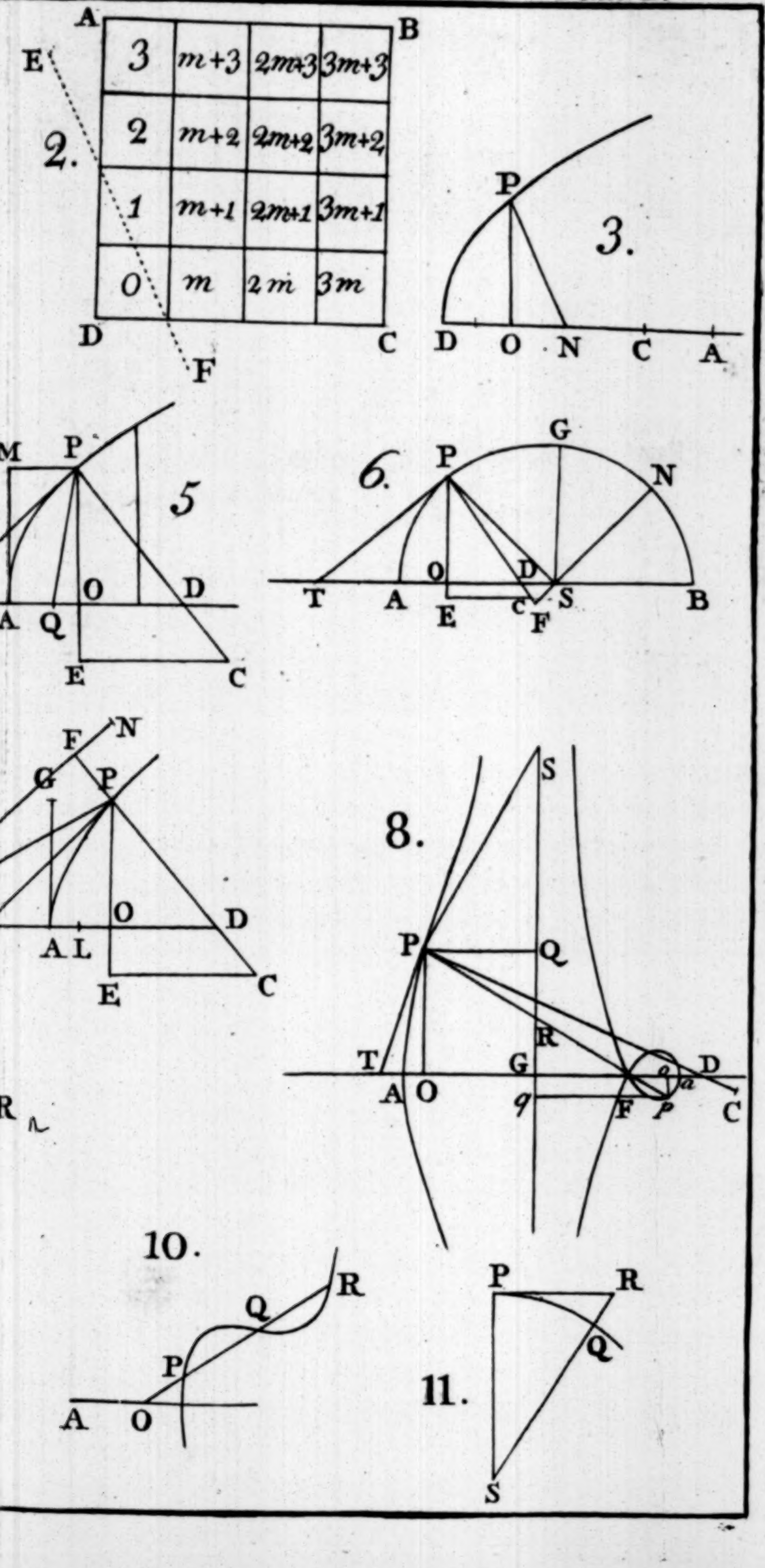
2.
A
B
E
3
m+3
2m+3
3m+3
2
m+2
2m+2
3m+2
1
m+1
2m+1
3m+1
0
m
2m
3m
D
C
F
3.
P
D
O
N
C
A
5
M
P
A
Q
O
D
E
C
6.
G
P
N
T
A
O
E
c
D
F
S
B
F
N
G
P
A
L
O
D
E
C
8.
S
P
Q
R
T
G
D
A
O
q
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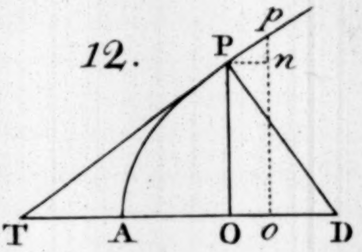
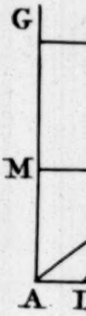
a
C
R
10.
R
Q
P
A
O
P
R
Q
11.
S

12.





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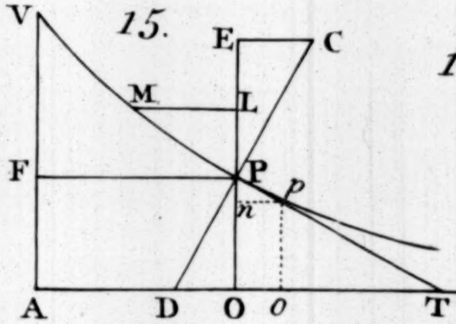

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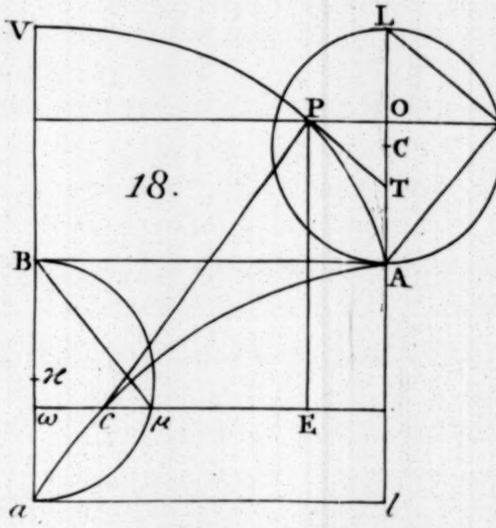

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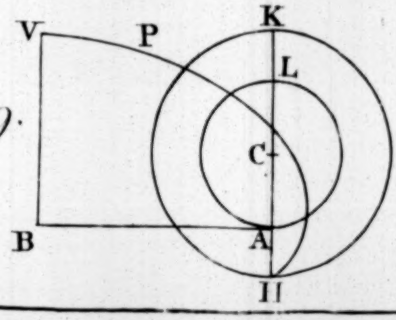

Plate II.

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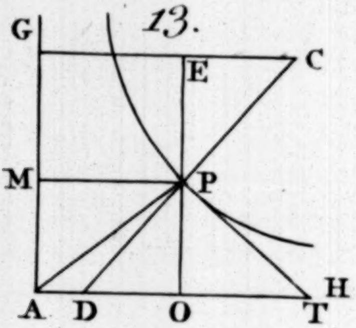

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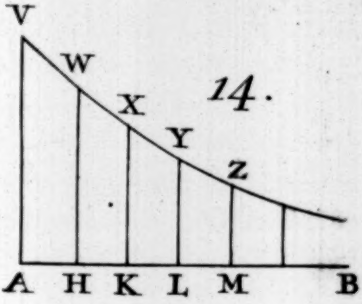

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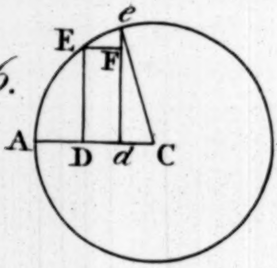

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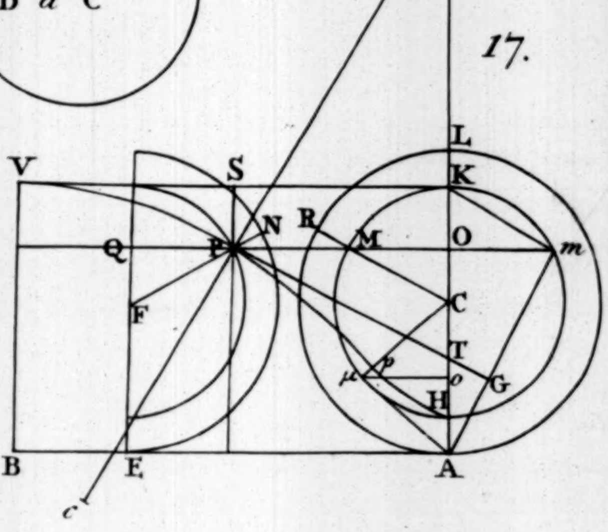

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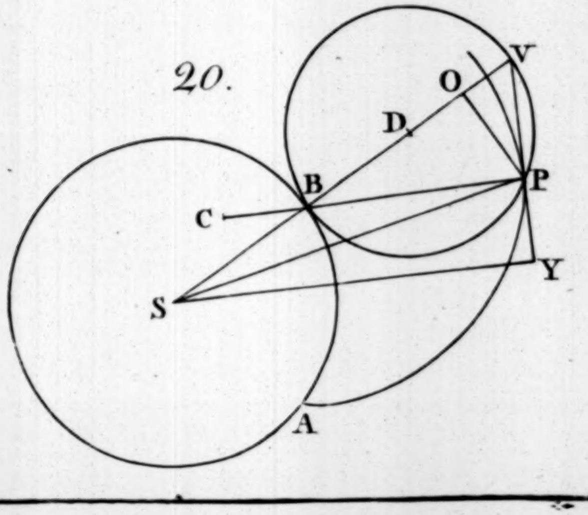

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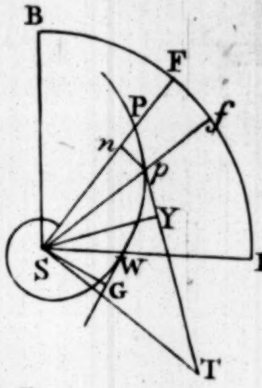

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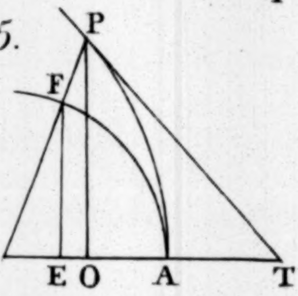

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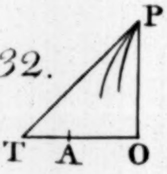

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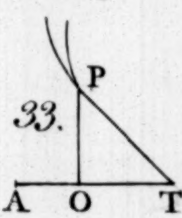

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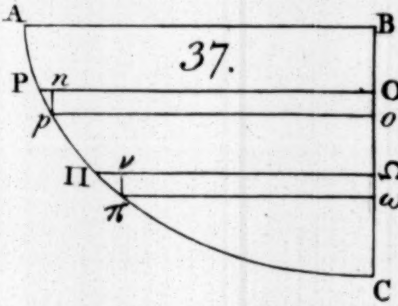

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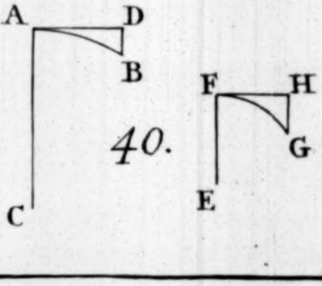
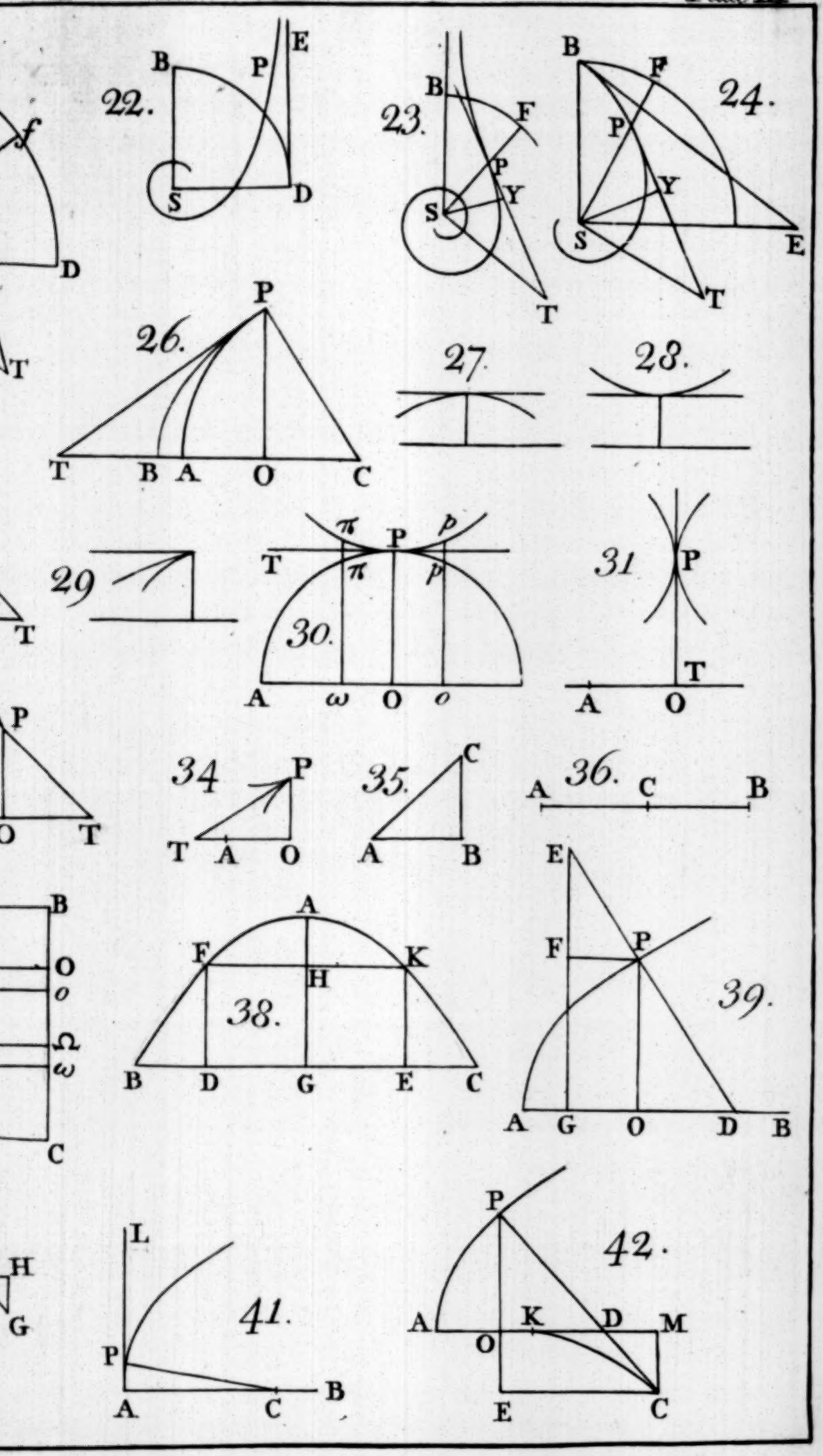
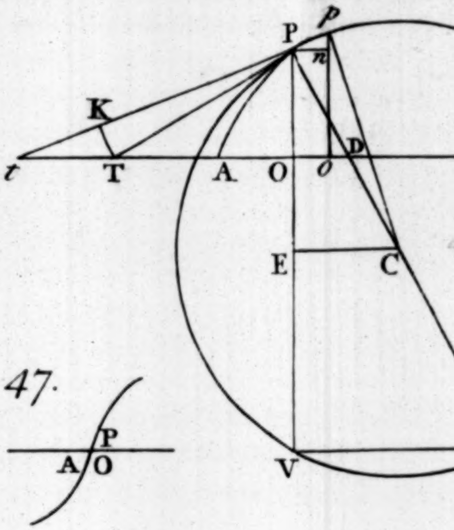


Plate III



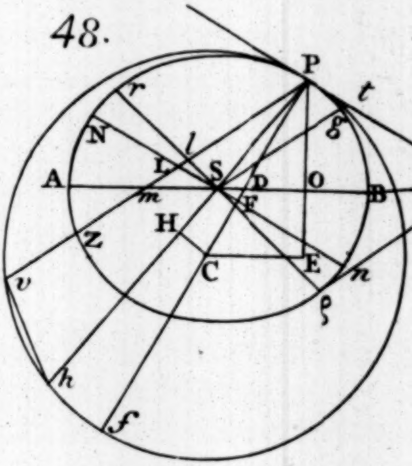
22.
B P E S D
23.
B F P Y S T
24.
B F P Y S E T
26.
P T B A O C
27.
28.
29
30.
T P π π p p A ω O o
31
P T A O
34
P T A O
35.
C A B
36.
A C B
38.
A F K H B D G E C
39.
E F P A G O D B
41.
L P A C B
42.
P A K D M O E C



P
p
n
K
D
t
T
A
O
o
E
C
V
P
A
O

47.

48.



P
t
r
N
l
L
S
D
O
A
m
F
B
H
Z
C
E
n
v
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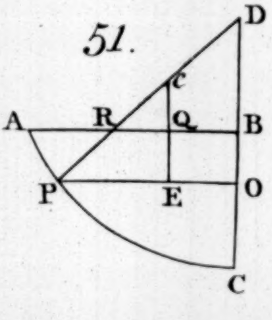
f

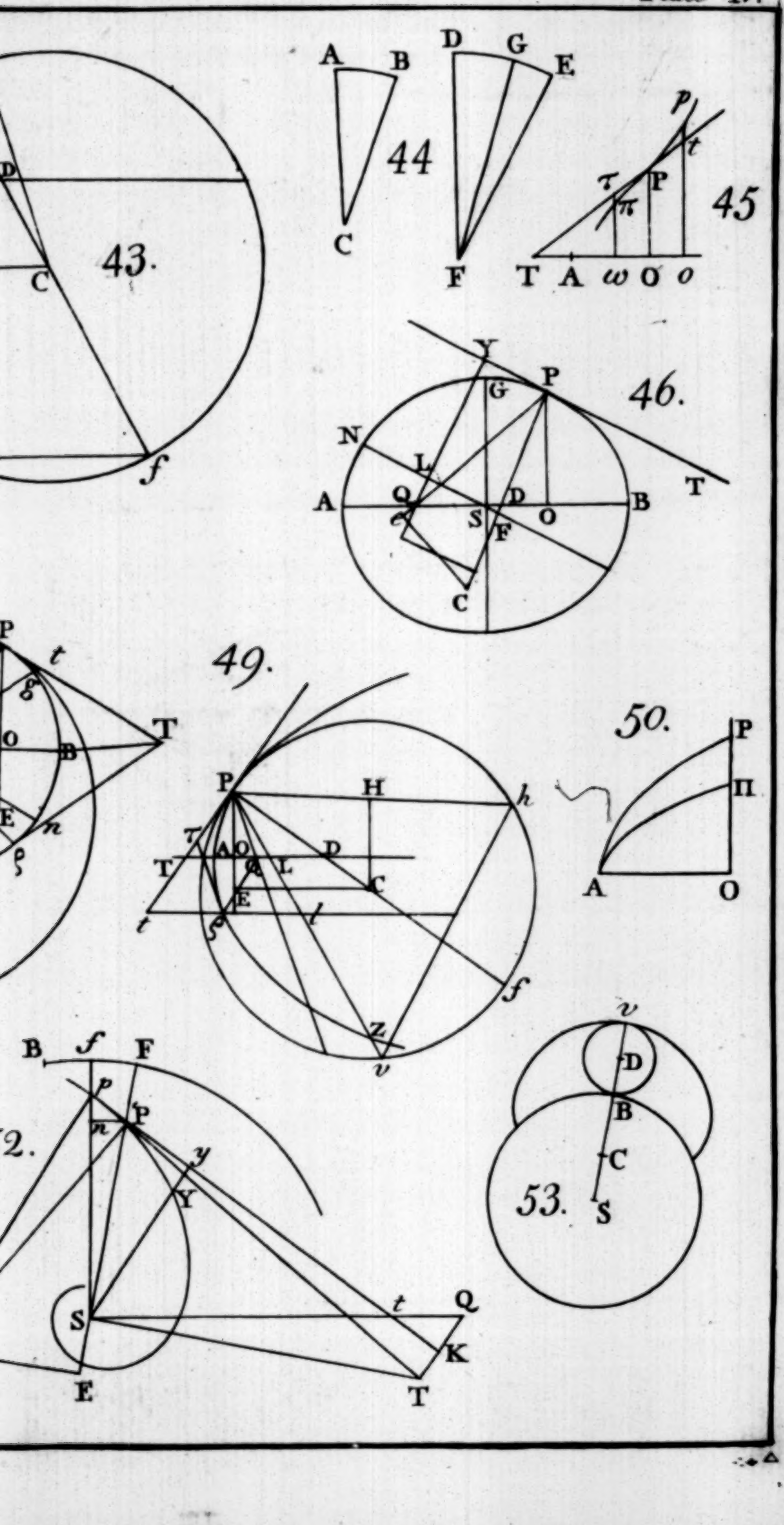
51.



D
c
A
R
Q
B
P
E
O
C

52.

B
f
S
C
E


43.
D
C
f
A B
C
D G E
44
F
p
t
τ
P
π
45
T A ω O o
Y
P
G
46.
N
L
T
A
Q
D
S
B
O
e
F
C
P
t
T
O
B
E
n
49.
P
H
h
τ
A O
D
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Q
L
C
E
t
l
f
Z
v
50.
P
Π
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O
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K
E
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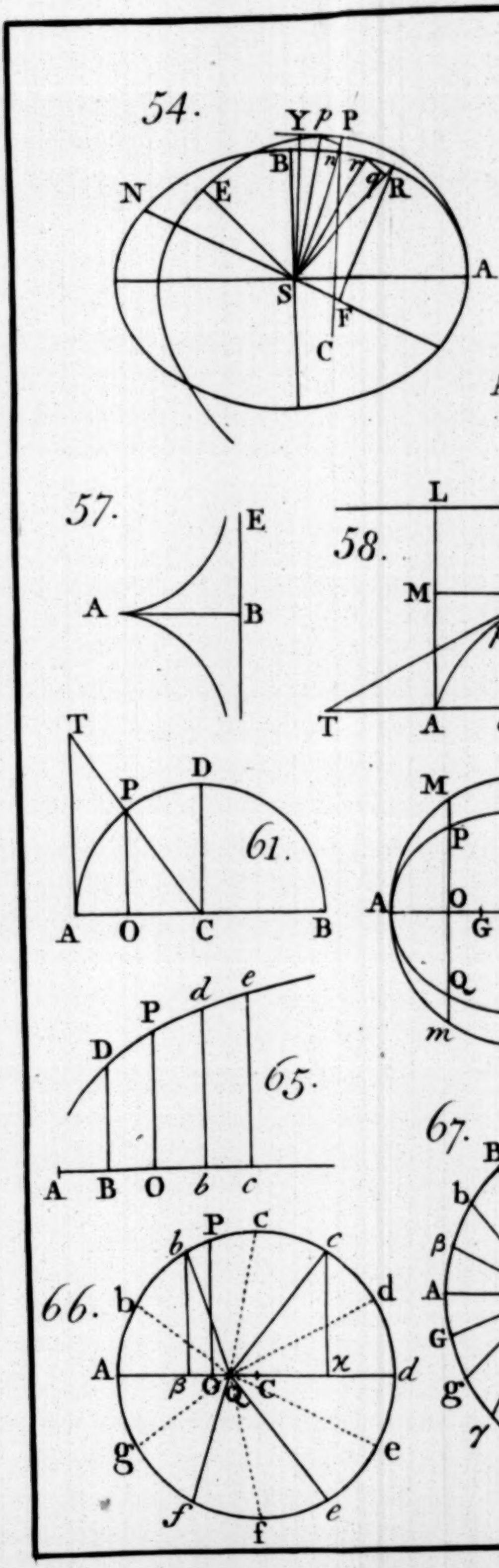
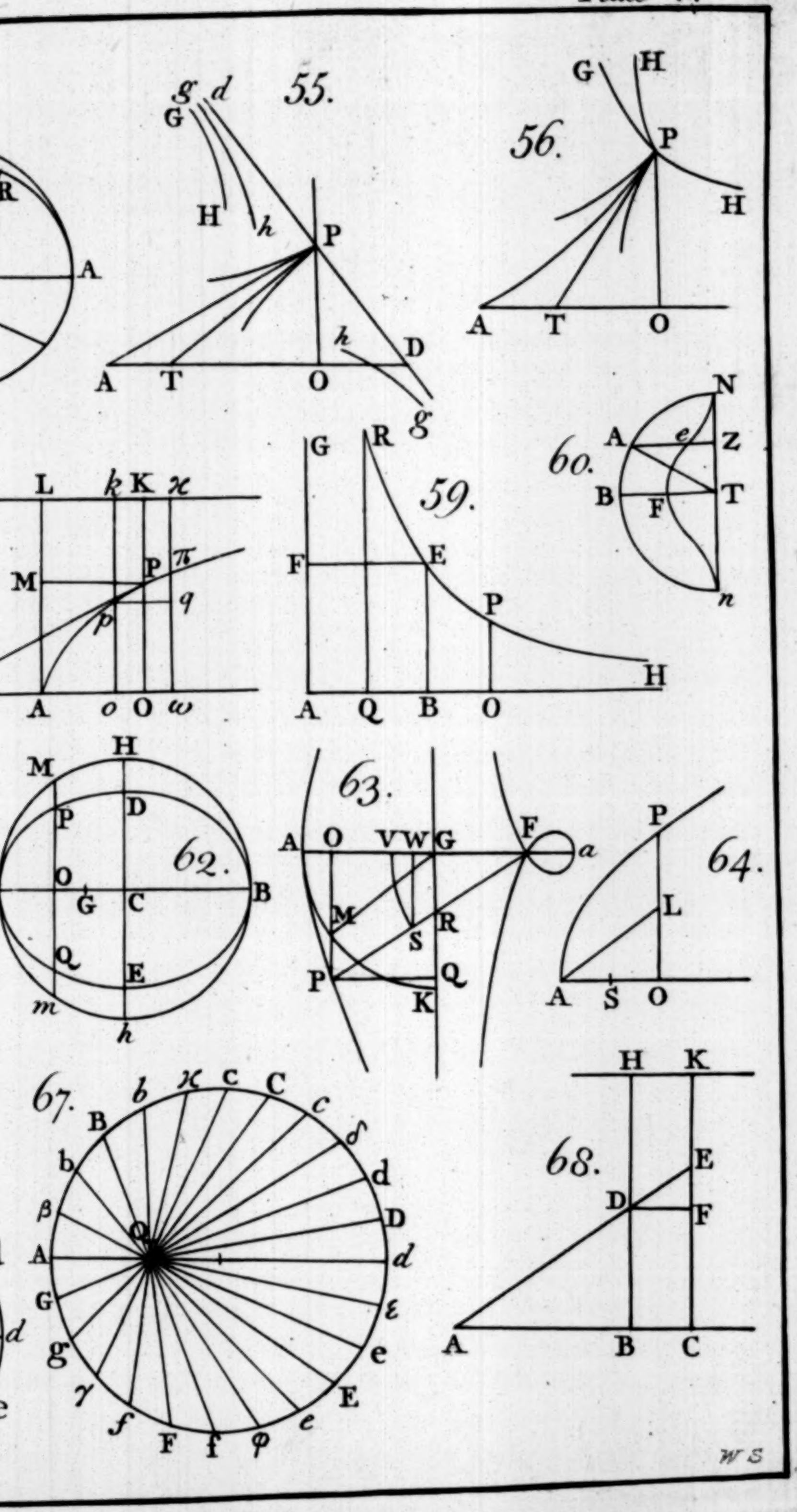

54.
57.
58.
61.
65.
66.
67.

Plate V.


55.
g d G H h P A T O h D g
56.
G H P H A T O
59.
G R F E P H A Q B O
60.
N A e Z B F T n
L k K ϰ M P π q p A o O ω
62.
M H P D O G C B Q E m h
63.
A O V W G F a M R S P Q K
64.
P L A S O
67.
b ϰ c C B c δ b d β D A d G ε g e γ E f F f φ e
68.
H K E D F A B C
W S

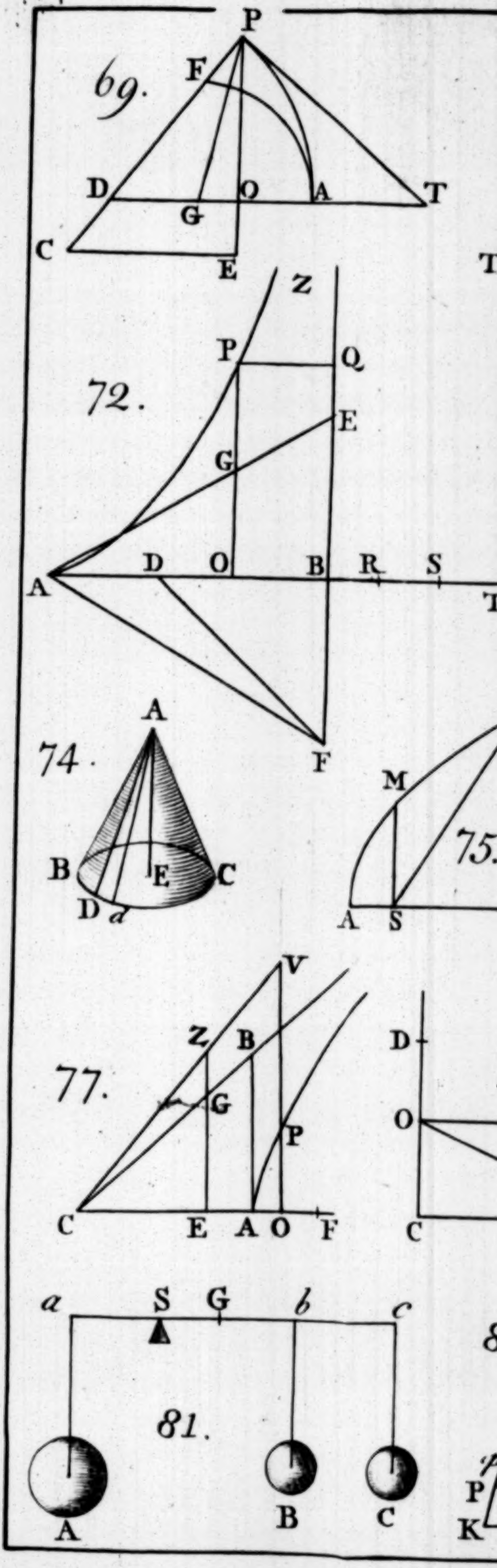
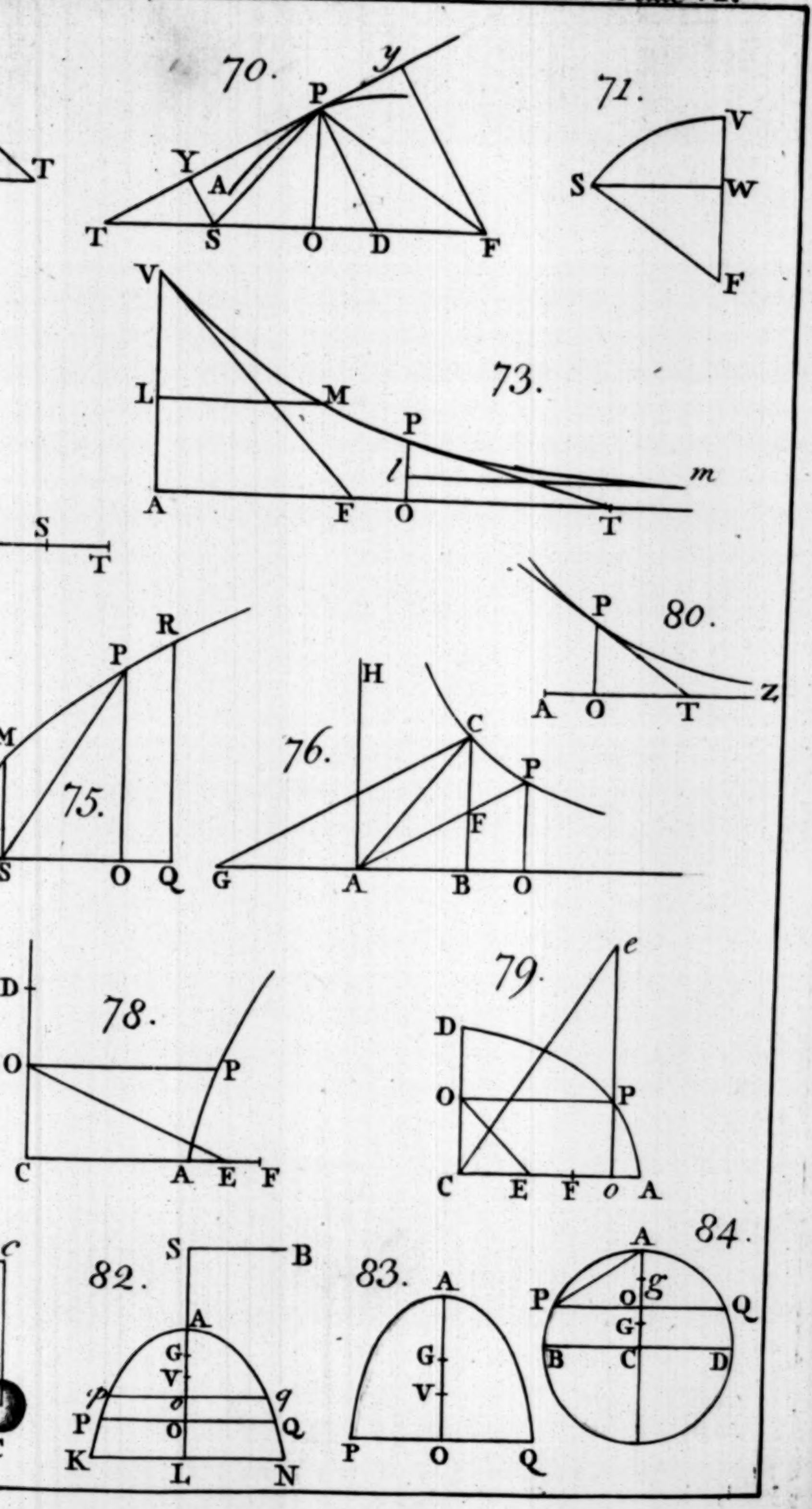

69.
72.
74.
75.
77.
81.

Plate VI.



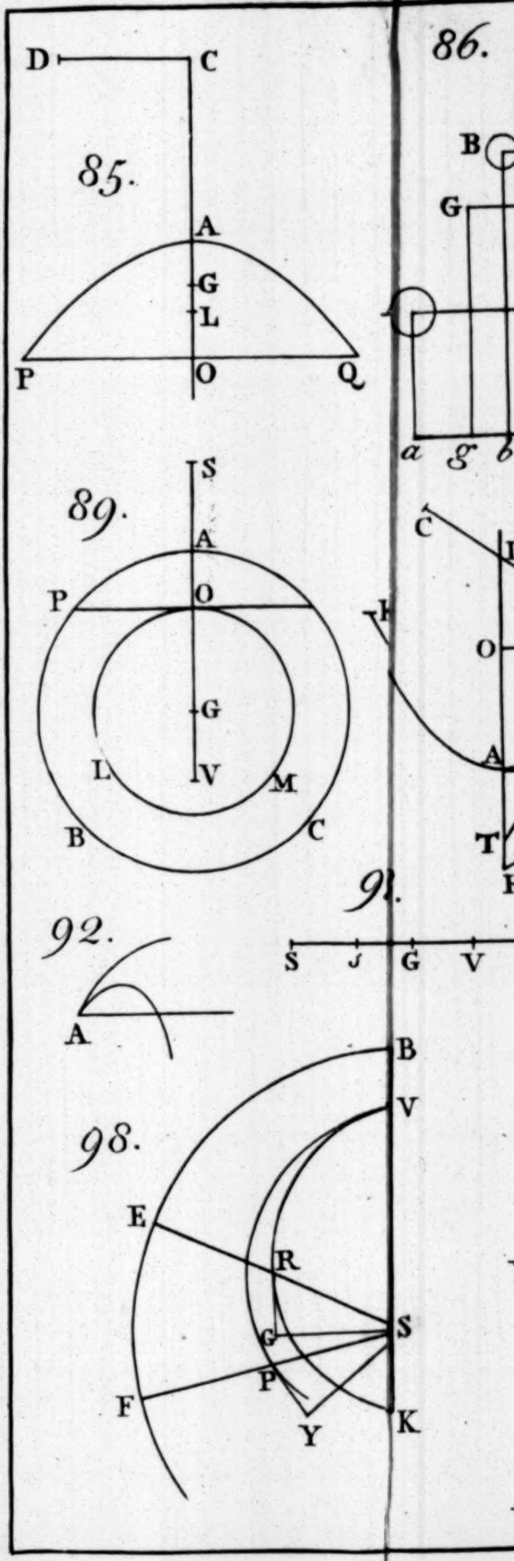
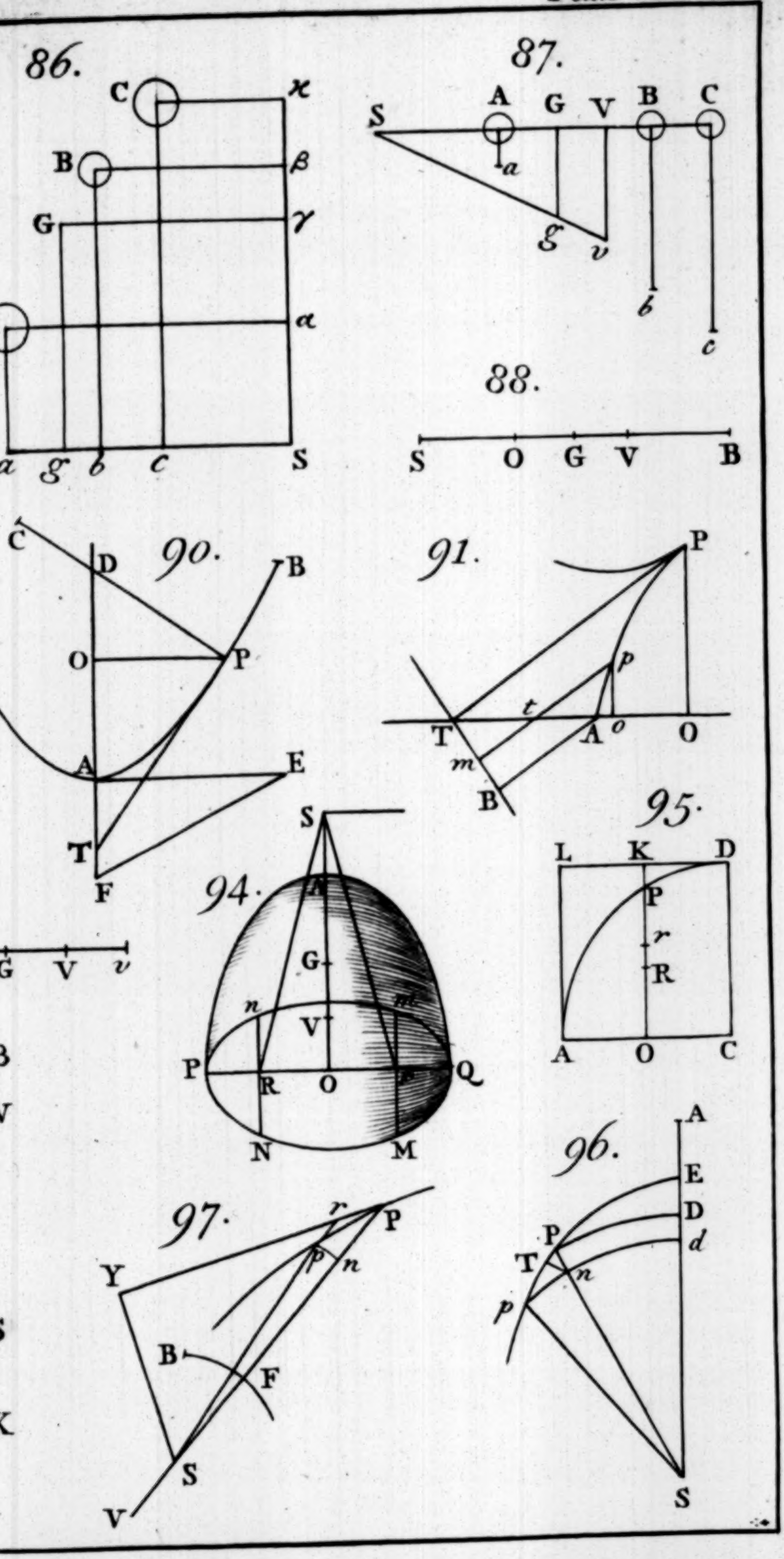

85.
D C A G L P O Q
86.
B G a g b
89.
S A P O G L V M B C
C O A T
91.
S s G V
92.
A
98.
B V E R S G P F Y K

Plate VII.


86.
87.
88.
90.
91.
94.
95.
96.
97.
C B G a g b c S x β γ α
S A G V B C a g v b c
S O G V B
C D O P B A E T F
P p t T A o O m B
S G V n m P R O Q N M
L K D P r R A O C
A E D d P T n p S
r P p n Y B F S V